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## SUMMARY

The fundamental equations for the plastic buckling of a rectangular plate under edge thrusts are developed on the basis of a new set of stress-strain relations for the behavior of a metal in the plastic range. These relations are derived for buckling from a state of uniform compression. The fundamental equation for the buckling of a simply compressed plate together with typical boundary conditions is then developed and the results are applied to calculating the buckling loads of a thin strip, a simply supported plate, and a cruciform section. Comparisons with the theories of Timoshenko and Ilyushin are made. Finally, an energy method is given which can be used for finding approximate values of the critical load.


## INTRODUCTION

This paper is concerned with the plastic buckling of a rectangular plate which, previous to buckling, is under a uniform compressive stress $\sigma_o$ in the direction of one of its edges. In the case of elastic buckling, in which $\sigma_o$ remains below the elastic limit of the plate material, it is well known that the buckling stress depends on the dimensions of the plate and on the manner in which it is supported (cf. reference 1, ch. 7). In the case of the plastic buckling of beams, on the other hand, Engesser (reference 2) and Von Kármán (reference 3) developed a satisfactory theory based on the fact that for a fiber which is compressed beyond the elastic limit the tangent modulus (i.e., the ratio of the variation of strain to the corresponding variation of stress) assumes different values depending on whether the variation of stress constitutes an increase or a relief of the existing compressive stress.

Generalization of this theory to the plastic buckling of plates has repeatedly been attempted. These attempts can be divided into two groups which may be labeled formal and analytical generalizations. The formal generalizations start from the remark that the formulas of the Engesser-Von Kármán theory of the plastic buckling of beams differ from the well-known formulas for the elastic buckling of beams only by the fact that the so-called "reduced modulus" replaces Young's modulus. A formal generalization of the Engesser-Von Kármán theory to the plastic buckling of plates is therefore obtained by introducing the reduced modulus into the formulas for the elastic buckling of plates in such a manner that the results of the Engesser-Von Kármán theory are obtained in the case of a narrow rectangular strip which is free on its long edges and simply supported on the short edges where it carries a compressive load. Of course, this formal generalization is more or less arbitrary and leads by no means to a unique result. Formulas of this type have been suggested by Bleich (reference 4, p. 216 ff.) and Timoshenko (reference 1, p. 384).

In contrast with these formal generalizations of the Engesser-Von Kármán theory, the analytical generalizations do not merely introduce the reduced modulus of the theory of beams into the formulas for the elastic buckling of plates. Instead, the analytical generalizations go back to the considerations by which the reduced modulus is derived and try to apply these to the case of a buckled plate. Generalizations of this kind have been previously presented by Kaufmann (reference 5) and Ilyushin (reference 6). As is shown in the present report, however, these authors use stress-strain relations which do not fulfill certain postulates of the theory of plasticity; the correctness of their results must therefore be questioned.

The present paper aims at developing a theory of the plastic buckling of plates which takes full account of the modern theory of plasticity. The stress-strain relations in the plastic range are discussed at considerable length in the first section of the ANALYSIS, and it is shown that,for an adequate treatment of buckling phenomena, a theory of plastic flow is indicated rather than a theory of plastic deformation of the type used by Kaufmann and Ilyushin. The precise definitions of these terms and the basic considerations suggesting the use of a theory of plastic deformation for problems such as buckling are fully discussed in the ANALYSIS. A particular theory of plastic flow suitable for the treatment of the problems under consideration is developed in the first section and its relations with other theories of plasticity are pointed out. It is shown that in the particular case of a plate buckling out of a state of simple compression there is very little freedom in the choice of the stress-strain relation if it is to fulfill certain simple postulates. This means that all the empirical information which is necessary for the theoretical treatment of the plastic buckling of a rectangular plate under edge thrusts can be obtained by a simple compression test.

The second section presents the development of the fundamental equation of the plastic buckling of a simply compressed plate, and the appropriate equations describing typical boundary conditions are given in the third section. The remaining parts contain several examples, which are carried out in detail, as well as an equivalent energy principle which proves to be very useful for approximate computations. Finally, the appendixes contain detailed discussions of several technical points raised in earlier parts of the paper.

The authors are indebted to Professor H. S. Tsien for drawing their attention to the work of Kaufmann (reference 5).

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## SYMBOLS

$a$ length of plate

$A$ constant in compressive stress-strain law

$\bar{A}$ expression in variational principle

$a', a'', a''', b', b'', b''', c'$ coefficients in plastic stress-strain law

$b$ width of plate

$B$ expression in variational principle

$c = (\lambda - 1)\Big/\left[(5 - 4\nu)\lambda - (1 - 2\nu)^2\right]$

$c_1, c_2, c_3, c_4$ arbitrary constants appearing in equation for $\dot{w}$

$D$ flexural rigidity of plate $\left(\frac{1}{12}h^3E_o\Big/\left(1 - \nu^2\right)\right)$

$D_{11}, D_{12}, D_{22}$ coefficients in plate equation for plastic flow

$D_{11}' = D_{11}/D, \quad D_{12}' = D_{12}/D, \quad D_{22}' = D_{22}/D$

$D_{11}^*, D_{12}^*, D_{22}^*$ coefficients in plate equation for Ilyushin's theory of plastic deformation

$\bar{D}_{11} = D_{11}^*/D, \quad \bar{D}_{12} = D_{12}^*/D, \quad \bar{D}_{22} = D_{22}^*/D$

$E$ tangent modulus in compression

| | |
|---|---|
| $E_o$ | Young's modulus |
| $E^*$ | Von Kármán's reduced modulus |
| $E_s$ | secant modulus obtained from compressive stress-strain diagram |
| $f(y)$ | section of buckled middle surface for $x = \text{Constant}$ |
| $h$ | thickness of plate |
| $I$ | moment of inertia of cross section |

$$k = 12\frac{\sigma_o a^2\left(1 - \nu^2\right)}{\pi^2 h^2 E_o}$$

$$\bar{k} = \frac{b^2}{a^2}k$$

$$\dot{K} = (2 - \nu)\dot{K}_1 + (2\nu - 1)\dot{K}_2$$

$$\dot{K}_1 = \partial^2\dot{w}/\partial x^2, \quad \dot{K}_2 = \partial^2\dot{w}/\partial y^2, \quad \dot{K}_{12} = 2\partial^2\dot{w}/\partial x\partial y$$

| | |
|---|---|
| $m$ | number of half waves in buckled configuration |
| $\dot{M}_x$ | rate of change of bending moment about y-axis |
| $\dot{M}_y$ | rate of change of bending moment about x-axis |
| $\dot{M}_{xy}$ | rate of change of twisting moment |
| $n$ | integer |
| $N_x$ | reduced compressive stress resultant $\left(\sigma_o h/E_o\right)$ |
| $\dot{N}_x$ | rate of change of stress resultant in x-direction |
| $\dot{N}_y$ | rate of change of stress resultant in y-direction |

$$p^2 = \left(\frac{m\pi}{a}\right)^2 \frac{D_{12}}{D_{22}}$$

| | |
|---|---|
| $P$ | total compressive force $(\sigma_o bh)$ |

$Q = \left[D_{12} - (1 - \nu)D\right] / D_{22}$

$q^2 = \frac{\sigma_o h}{D_{12}} - \left(\frac{m\pi}{a}\right)^2 \frac{D_{11}D_{22} - D_{12}^2}{D_{12}D_{22}}$

$r = \sqrt{p(q + p)}$

R side ratio (b/a)

$s = \sqrt{p(q - p)}$

t time

$u = 8\nu^2 + 12\nu - 23$

$\dot{w}$ deflection rate

x, y, z rectangular Cartesian coordinates; x,y-plane coincides with middle surface of unbuckled plate

$z_o = \frac{(2 - \nu)\dot{\epsilon}_1 + (2\nu - 1)\dot{\epsilon}_2}{\dot{K}}$

$\alpha = 1 - \frac{2}{c(5 - 4\nu)}$

$\alpha$ ratio of Von Kármán's modulus to Young's modulus (in section "Buckling of a simply supported plate" only)

$\beta$ constant in compressive stress-strain law

$$\delta = \begin{cases} \frac{1}{2}\left[1 - \frac{3}{2}\zeta_o^+ + \frac{1}{2}\left(\zeta_o^+\right)^3\right] & \text{for } \dot{K} > 0 \\ \frac{1}{2}\left[1 + \frac{3}{2}\zeta_o^- - \frac{1}{2}\left(\zeta_o^-\right)^3\right] & \text{for } \dot{K} < 0 \end{cases}$$

$\Delta = \frac{D_{11}'D_{22}' - D_{12}'^2}{D_{22}'^2}$

$\epsilon$ uniaxial strain

$d\epsilon_x$, $d\epsilon_y$, $d\epsilon_z$, $d\gamma_{xy}$, $d\gamma_{yz}$, $d\gamma_{zx}$ infinitesimal strain increments present in buckling

$d\epsilon_x'$, $d\epsilon_y'$, $d\epsilon_z'$, $d\gamma_{xy}'$ reversible (elastic) strain increments

$d\epsilon_x''$, $d\epsilon_y''$, $d\epsilon_z''$, $d\gamma_{xy}''$ permanent (plastic) strain increments

$\dot{\epsilon}_1(x,y)$ normal strain rate in middle surface in x-direction

$\dot{\epsilon}_2(x,y)$ normal strain rate in middle surface in y-direction

$\dot{\gamma}(x,y)$ shear strain rate in middle surface

$\zeta_o = 2z_o/h$

$\zeta_o^+ = \alpha + \sqrt{\alpha^2 - 1}$

$\zeta_o^- = -\alpha - \sqrt{\alpha^2 - 1}$

$\eta = sb$

$\kappa$ ratio of Von Kármán's modulus to Young's modulus $(E^*/E_o)$

$\lambda$ ratio of Young's modulus to tangent modulus $(E_o/E)$

$\nu$ Poisson's ratio

$\xi = rb$

$\sigma$ uniaxial stress

$\sigma_i$ intensity of stress $\left(\sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2}\right)$

$\sigma_{cr}$ critical compressive stress

$\sigma_o = -\sigma_x$ original compressive stress in plate

$\sigma_x$, $\sigma_y$, $\sigma_z$ normal stress components

$\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$ shear stress components

$d\sigma_x$, $d\sigma_y$, $d\sigma_z$, $d\tau_{xy}$, $d\tau_{yz}$, $d\tau_{zx}$ infinitesimal stress increments present in buckling

$$\psi = \omega\left(1 - \frac{1}{2}\sqrt{\kappa}\right)\left[\left(1 - \frac{1}{2}\sqrt{\kappa}\right)^2 + \frac{3}{4}\,\frac{\kappa}{1 - \left(1 - \frac{1}{2}\sqrt{\kappa}\,\omega\right)}\right]$$

$\omega$ function of the intensity of stress $\sigma_i$

$\omega' = d\omega/d\sigma_i$

$\Omega = \omega/(1 - \omega)$

$\dot{\Omega} = \dfrac{d\Omega}{d\sigma_i}\,\dot{\sigma}_i$

$\Omega' = d\Omega/d\sigma_i$

$-$ superscript denoting values on unloading side of neutral surface

$+$ superscript denoting values on loading side of neutral surface

## ANALYSIS

### Stress-Strain Relations for Buckling from a State of Uniform Compression

The mechanism of buckling beyond the elastic limit is relatively complicated because the material, which was originally in a state of simple compression, is loaded in some regions and unloaded in others during the buckling process. Consequently, the stress-strain relations must be considered in some detail with special reference to the problem of loading beyond the elastic limit followed by unloading.

The material must exhibit strain-hardening if the determination of the buckling stress is to constitute a problem. Indeed, for a perfectly plastic material which yields under constant stress, Von Kármán's reduced modulus vanishes once the initial compressive stress has reached the yield limit. This means that the bending stiffness is reduced to zero and buckling must be expected quite independent of the dimensions of the bar.

Stress-strain laws for materials which exhibit strain-hardening can be divided into two types which, for convenience, will be called "theories of plastic deformation" and "theories of plastic flow." According to the first group, there exists a one-to-one correspondence between stress and strain in the plastic range, as well as the elastic, provided that the material is being loaded. The stress-strain law of the well-known Hencky-Nádai theory (reference 7, ch. 14, and reference 8) and the law used by Ilyushin (reference 6) in his discussion of plastic buckling are typical theories of plastic deformation. On the other hand, the theories of plastic flow are based on the assumption that, for a given state of stress, there exists a one-to-one correspondence between the rates of change of stress and strain in such a manner that the resulting relation between stress and strain cannot be integrated so as to yield a relation between stress and strain alone. Typical examples of theories of plastic flow are the stress-strain relations developed by Prager (reference 9) and Handelman, Lin, and Prager (reference 10). A particularly important difference between these two basic theories of plasticity lies in the fact that according to the theory of plastic deformation the strain which corresponds to a certain state of stress the theory of plastic deformation is entirely independent of the manner in which this state of stress has been reached, whereas, according to the theory of plastic flow, the strain depends on the manner in which the state of stress is built up.

The stress-strain relations to be used in the analysis of the plastic buckling of a rectangular plate under edge thrusts form a special case of those developed by Handelman, Lin, and Prager in reference 10. In this particular case, however, it is possible to develop the stress-strain relation in a quite elementary manner, and the inherent difficulties of the theories of plastic deformation can be seen from a slightly different point of view. It appears worth while, then, to examine these relations in some detail with special reference to the problem which forms the subject of the present report.

In the following, the stresses and strains in the buckled plate will be referred to a fixed system of rectangular Cartesian coordinates $x$, $y$, and $z$. The $x,y$-plane of this coordinate system coincides with the middle surface of the unbuckled plate, and the axes of $x$ and $y$ coincide with two of its edges, the other edges falling on the lines $x = a$ and $y = b$. Prior to buckling, the plate is under a uniform compressive stress $\sigma_o$ in the direction of the x-axis (fig. 1). The investigation of the stability of the state of stress

$$\left.\begin{aligned} \sigma_x &= -\sigma_o \\ \sigma_y &= \sigma_z = 0 \\ \tau_{xy} &= \tau_{yz} = \tau_{zx} = 0 \end{aligned}\right\} \qquad (1)$$

requires the knowledge of the relations between the infinitesimal increments of stress $d\sigma_x$, $d\sigma_y$, $d\sigma_z$, $d\tau_{xy}$, $d\tau_{yz}$, and $d\tau_{zx}$ and the corresponding increments of strain $d\epsilon_x$, $d\epsilon_y$, $d\epsilon_z$, $d\gamma_{xy}$, $d\gamma_{yz}$, and $d\gamma_{zx}$. Within the framework of plate theory, $\sigma_z = \tau_{xz} = \tau_{yz} = 0$, even in the buckled state, and hence $d\sigma_z = d\tau_{xz} = d\tau_{yz} = 0$. Accordingly, $d\gamma_{xz} = d\gamma_{yz} = 0$. Within the elastic range the remaining increments of stress and strain are related to each other by means of

$$\left.\begin{aligned}
E_o\, d\epsilon_x &= d\sigma_x - \nu\, d\sigma_y \\
E_o\, d\epsilon_y &= -\nu\, d\sigma_x + d\sigma_y \\
E_o\, d\epsilon_z &= -\nu\, d\sigma_x - \nu\, d\sigma_y \\
E_o\, d\gamma_{xy} &= 2(1 + \nu)\, d\tau_{xy}
\end{aligned}\right\} \qquad (2)$$

where $E_o$ denotes Young's modulus and $\nu$, Poisson's ratio. Before an analysis of the plastic buckling of the plate can be attempted, the relations replacing equations (2) in the plastic range must be known. In order to establish these relations, it will be convenient to think of the strain increments as consisting of reversible (elastic) and permanent (plastic) components:

$$\left.\begin{aligned}
d\epsilon_x &= d\epsilon_x' + d\epsilon_x'' \\
d\epsilon_y &= d\epsilon_y' + d\epsilon_y'' \\
d\epsilon_z &= d\epsilon_z' + d\epsilon_z'' \\
d\gamma_{xy} &= d\gamma_{xy}' + d\gamma_{xy}''
\end{aligned}\right\} \qquad (3)$$

Primes and double primes denote elastic and plastic components, respectively. The elastic increments of strain are related to the increments of stress by means of equations (2), in which the left-hand sides must all be written with primes now:

$$\left.\begin{aligned}
E_o\, d\epsilon_x{}' &= d\sigma_x - \nu\, d\sigma_y \\
E_o\, d\epsilon_y{}' &= -\nu\, d\sigma_x + d\sigma_y \\
E_o\, d\epsilon_z{}' &= -\nu\, d\sigma_x - \nu\, d\sigma_y \\
E_o\, d\gamma_{xy}{}' &= 2(1 + \nu)\, d\tau_{xy}
\end{aligned}\right\} \quad (4)$$

These relations may be regarded as the definitions of the elastic increments of strain. The purpose of the following discussion is to establish similar relations for the plastic increments of strain.

The elastic increments of strain, equations (4), depend only on the increments of stress and are independent of the existing stress $\sigma_o$. Moreover, a reversal of the signs of all increments of stress leads to a mere reversal of the signs of all elastic increments of strain. The plastic increments of strain, however, do not have these properties; since they must vanish as long as $\sigma_o$ remains below the elastic limit, they cannot be independent of the existing stress $\sigma_o$. Moreover, if for a given value of $\sigma_o$ certain stress increments produce plastic increments of strain, stress increments of the same magnitudes but opposite signs do not produce any plastic deformation. In other terms, beyond the limit of elasticity an infinitesimal change of stress may be classified as loading the material or not according to whether it is accompanied by permanent deformation. Infinitesimal changes of stress which do not load the material may be classified in turn as unloading or neutral. Unloading brings the material into a state of stress such that all sufficiently small further changes of stress are accompanied by elastic deformations only. These basic differences in loading and unloading appear somewhat more clearly if the simple example of a uniaxial state of stress and strain (say a tensile test) is considered. Let $\sigma$ denote the stress and $\epsilon$ the strain in figure 2 and suppose the material is loaded to the point P. The stress-strain diagram for unloading is a straight line PA with the same slope as the loading curve at the origin O. The permanent strain corresponding to loading up to the point P is measured by OA. Suppose now that, after the point P has been reached, the test specimen is further loaded to the point $P_1$; by this, the permanent strain is increased by the amount $AA_1$. In other words, the change from P to $P_1$ constitutes loading in the sense just defined. On the other hand, if the new state of stress and strain is given by the point $P_2$, that is, if the stress has been reduced below that at P, the permanent strain is left unchanged. Furthermore, any small change of stress from the point $P_2$ (strictly, all changes within the ranges $P_2P$ and $P_2A$) produces an additional deformation which is purely elastic. The material has thus been unloaded.

For uniaxial stress any change of stress constitutes either loading or unloading. A third possibility, designated as "neutral" change of stress, exists in the case of combined stress. A neutral change of stress, while not accompanied by a permanent deformation, brings the material into a state such that there exist certain further changes of stress which are arbitrarily small and yet produce a permanent deformation. This third condition is illustrated in the analysis of the buckling of a plate. It is precisely the possibility of the occurrence of neutral changes of stress which distinguishes the present problem from that treated by Engesser and Von Kármán, for in their case the stress is uniaxial in the buckled state as well as in the unbuckled. Accordingly, a change of stress can be only an increase of the existing compressive stress (loading) or a decrease (unloading). The situation is more complicated in the case of a plate.

Since there is no permanent deformation accompanying neutral changes of stress or unloading, $d\epsilon_x'' = d\epsilon_y'' = d\epsilon_z'' = d\gamma_{xy}'' = 0$; and the relations of equations (4) define the total change of strain. For loading, however, equations (4) must be supplemented by equations of the form

$$\left.\begin{aligned}
E_o\, d\epsilon_x'' &= a'\, d\sigma_x + b'\, d\sigma_y \\
E_o\, d\epsilon_y'' &= a''\, d\sigma_x + b''\, d\sigma_y \\
E_o\, d\epsilon_z'' &= a'''\, d\sigma_x + b'''\, d\sigma_y \\
E_o\, d\gamma_{xy}'' &= 2c'\, d\tau_{xy}
\end{aligned}\right\} \qquad (5)$$

where the coefficients $a'$, $b'$, $a''$, $b''$, $a'''$, $b'''$, and $c'$ depend on the existing stress $\sigma_o$.

As is customary in the theory of plasticity, the plastic deformations will be supposed to represent a mere change in shape but no change in volume. Accordingly,

$$d\epsilon_x'' + d\epsilon_y'' + d\epsilon_z'' = 0 \tag{6}$$

This relation must hold independently of the values of $d\sigma_x$ and $d\sigma_y$. Thus,

$$a' + a'' + a''' = 0 \tag{7}$$

and

$$b' + b'' + b''' = 0 \tag{8}$$

The elastic formulas, equations (4), exhibit a certain symmetry of the coefficients appearing on the right-hand side. For instance, the coefficients of $d\sigma_x$ in the second and third equations are equal, as are the coefficients of $d\sigma_y$ in the first and $d\sigma_x$ in the second equations. Which of these symmetries, if any, will be maintained in equations (5)? The existing state of stress singles out the x-axis, but it does not matter which of the other two axes is labeled $y$ and which $z$. Accordingly,

$$a'' = a''' \tag{9}$$

In view of equations (7) and (9),

$$a'' = a''' = -\frac{1}{2}a' \tag{10}$$

These coefficients can easily be expressed in terms of the so-called "tangent modulus" corresponding to the compressive stress $\sigma_o$. Application of equations (4) and (5) to simple compression in the x-direction yields (with $d\sigma_y = 0$)

$$\begin{aligned} E_o\, d\epsilon_x &= E_o\left(d\epsilon_x' + d\epsilon_x''\right) \\ &= d\sigma_x + a'\, d\sigma_x \end{aligned} \tag{11}$$

or

$$E = \frac{d\sigma_x}{d\epsilon_x} = \frac{E_o}{1 + a'} \tag{12}$$

where $E$ denotes the tangent modulus. With

$$\lambda = E_0/E \tag{13}$$

equation (11) gives

$$a' = \lambda - 1 \tag{14}$$

Hence, according to equation (10),

$$a'' = a''' = \frac{-(\lambda - 1)}{2} \tag{15}$$

Next, the criterion for neutral changes of stress must be considered. Any given infinitesimal change of strain can be decomposed in the following manner:

$$\left.\begin{aligned}
d\epsilon_x &= \frac{1}{3}\left(d\epsilon_x + d\epsilon_y + d\epsilon_z\right) + \frac{1}{3}\left(2d\epsilon_x - d\epsilon_y - d\epsilon_z\right) \\
d\epsilon_y &= \frac{1}{3}\left(d\epsilon_x + d\epsilon_y + d\epsilon_z\right) + \frac{1}{3}\left(-d\epsilon_x + 2d\epsilon_y - d\epsilon_z\right) \\
d\epsilon_z &= \frac{1}{3}\left(d\epsilon_x + d\epsilon_y + d\epsilon_z\right) + \frac{1}{3}\left(-d\epsilon_x - d\epsilon_y + 2d\epsilon_z\right) \\
d\gamma_{xy} &= 0 + d\gamma_{xy} \\
d\gamma_{yz} &= 0 + d\gamma_{yz} \\
d\gamma_{zx} &= 0 + d\gamma_{zx}
\end{aligned}\right\} \tag{16}$$

The change of strain defined by the first members of the right-hand sides of these equations is a uniform expansion (or contraction) in all directions. Such a uniform expansion changes the volume but not the shape of the element to which it is applied. The change of strain defined by the second members of the right sides of equations (16), on the other hand, affects the shape of the element but preserves its volume. The work done by the existing stress or stresses on the change of strain, equations (16), consists of the work done on the change of volume represented by the first members of the right sides of equations (16) and the work done on the change of shape represented by the second members. Since all changes of volume are supposed to be of an elastic nature, it seems natural to speak of loading or unloading according to whether the work $dW$ which the existing stresses do on the change of shape alone is positive or negative. Vanishing of this work must then be interpreted as indicating a neutral change.

In the case under discussion, the only existing stress is $\sigma_x = -\sigma_o$, and the criterion for loading or unloading is furnished by the sign of

$$dW = -\frac{\sigma_o}{3}\left(2d\epsilon_x - d\epsilon_y - d\epsilon_z\right) \tag{17}$$

while neutral changes are characterized by

$$2d\epsilon_x - d\epsilon_y - d\epsilon_z = 0 \tag{18}$$

Now, for unloading, the entire change of strain is of an elastic nature and equations (2) apply. Equation (17) is therefore equivalent to

$$E_o\ dW = -\frac{\sigma_o}{3}\left[2\left(d\sigma_x - \nu\ d\sigma_y\right) - \left(-\nu\ d\sigma_x + d\sigma_y\right) - \left(-\nu\ d\sigma_x - \nu\ d\sigma_y\right)\right] \tag{19}$$

Since $\sigma_o > 0$, this expression will be negative, whenever

$$2d\sigma_x - d\sigma_y > 0 \tag{20}$$

This inequality, equation (20), is thus seen to constitute the criterion for unloading. Similarly, the criterion for neutral changes of stress is found to be

$$2d\sigma_x - d\sigma_y = 0 \tag{21}$$

Changes of stress which satisfy neither equation (20) nor (21), that is, changes of stress for which

$$2d\sigma_x - d\sigma_y < 0 \tag{22}$$

must therefore constitute loading. Another definition for the criterion for the three types of change of stress, which is found by combining equations (20), (21), and (22), is that the change of stress is classified by the sign of the increment in the second invariant of the stress deviator, which measures the intensity of stress. A detailed account of this alternative formulation is found in reference 10.

By a suitable choice of the values of $d\sigma_x$ and $d\sigma_y$, the expression $2d\sigma_x - d\sigma_y$ can be made to fulfill the following inequalities:

$$0 > 2d\sigma_x - d\sigma_y > -\epsilon \tag{23}$$

where $\epsilon$ is an arbitrarily prescribed small positive number. All changes of stress satisfying equation (23) constitute loading and are therefore accompanied by plastic deformations in accordance with equations (5). For $\epsilon \longrightarrow 0$, however, these changes tend toward neutral changes of stress for which there are no plastic deformations. Furthermore, there are no plastic increments of strain when $2d\sigma_x - d\sigma_y > 0$. It is to be expected that the total strain increments will be continuous in the region which marks the transition from unloading through the neutral state to loading. Although such a statement does not follow specifically from the equations of equilibrium or compatibility, continuity should be expected in the strain increments. With this assumption, the plastic increments of strain, equations (5), should vanish whenever the increments of stress satisfy equation (21). This furnishes the conditions

$$\left.\begin{aligned} a' + 2b' &= 0 \\ a'' + 2b'' &= 0 \\ a''' + 2b''' &= 0 \\ c' &= 0 \end{aligned}\right\} \tag{24}$$

Together with equations (14) and (15), these equations determine all coefficients appearing in equations (5), which therefore take the form

$$\left.\begin{aligned}
E_o\ d\epsilon_x'' &= (\lambda - 1)\ d\sigma_x - \frac{\lambda - 1}{2}\ d\sigma_y \\
E_o\ d\epsilon_y'' &= -\frac{\lambda - 1}{2}\ d\sigma_x + \frac{\lambda - 1}{4}\ d\sigma_y \\
E_o\ d\epsilon_z'' &= -\frac{\lambda - 1}{2}\ d\sigma_x + \frac{\lambda - 1}{4}\ d\sigma_y \\
E_o\ d\gamma_{xy}'' &= 0
\end{aligned}\right\} \tag{25}$$

It is interesting to note that here again the coefficients of $d\sigma_x$ in the second and third equations are equal, as are the coefficients of $d\sigma_y$ in the first and of $d\sigma_x$ in the second equation. Whereas in the elastic case this type of symmetry in the stress-strain relations is a consequence of the isotropy of the material, this is no longer so in the case of equations (25). Indeed, the equality of $a''$ and $a'''$ (see equation (15)) follows from the assumption that the plastic deformations do not involve a change in volume. The equality of $b'$ and $a''$, on the other hand, might be described as almost accidental, the value of the ratio $a''/a'$ being fixed by the assumption just mentioned, while the value of the ratio $b'/a'$ is fixed by the form of the condition for neutral change of stress.

Combination of equations (4) and (25) finally yields the stress-strain relations which will be used throughout this paper:

$$\left.\begin{aligned}
E_o\ d\epsilon_x &= \lambda\ d\sigma_x - \left(\nu + \frac{\lambda - 1}{2}\right) d\sigma_y \\
E_o\ d\epsilon_y &= -\left(\nu + \frac{\lambda - 1}{2}\right) d\sigma_x + \frac{\lambda + 3}{4}\ d\sigma_y \\
E_o\ d\epsilon_z &= -\left(\nu + \frac{\lambda - 1}{2}\right) d\sigma_x - \left(\nu - \frac{\lambda - 1}{4}\right) d\sigma_y \\
E_o\ d\gamma_{xy} &= 2(1 + \nu)\ d\tau_{xy}
\end{aligned}\right\} \tag{26}$$

It seems worth while to stress once again the assumptions on the basis of which these stress-strain relations are derived. These are

(1) Plastic deformations do not involve a change of volume

(2) The criterion for loading or unloading is furnished by the sign of the work $dW$ which the existing stresses do on the change of shape produced by the increments of stress

The first assumption is commonly made in the theory of plasticity (cf. reference 7, p. 10) and is confirmed by the experiments of Bridgman (reference 11, p. 166). The second assumption is a slight generalization of a similar assumption which Prager (reference 10) introduced in the case of incompressible plastic materials; more recently, it has been used by Ilyushin (reference 6).

It is interesting to note how far the stress-strain relations, equations (26), differ from those used in previous work on the plastic buckling of plates. In the present notation, Kaufmann's stress-strain relations (reference 5) are

$$\left.\begin{aligned} E_o\ d\epsilon_x &= \lambda\ d\sigma_x - \nu\ d\sigma_y \\ E_o\ d\epsilon_y &= -\lambda\nu\ d\sigma_x + d\sigma_y \\ E_o\ d\gamma_{xy} &= (1 + \lambda)(1 + \nu)\ d\tau_{xy} \end{aligned}\right\} \tag{27}$$

(The expression for $d\epsilon_z$ is not given because this strain component is not necessary for the determination of the bending and twisting moments in the buckled plate.) It is seen that here the coefficients of $d\sigma_y$ in the first equation and that of $d\sigma_x$ in the second are unequal. In an earlier paper on the plastic buckling of cylindrical shells (reference 12, footnote 1, p. 422) in which similar stress-strain relations were used, Kaufmann comments on this lack of symmetry, recommending that the stress-strain relations, equations (25), be checked by experiment. Since this type of symmetry in the present stress-strain relations, equations (26), has been characterized as almost accidental, the lack of symmetry in Kaufmann's relations hardly constitutes a sufficient reason for discarding the stress-strain relations, equations (27). It is not difficult, however, to show that these relations correspond to an unacceptable condition for neutral changes of stress. Indeed, subtraction of the elastic increments of strain, equations (4), from the total increments of strain, equations (27), yields the following plastic increments of strain:

$$\left.\begin{aligned} E_o\, d\epsilon_x'' &= (\lambda - 1)\ d\sigma_x \\ E_o\, d\epsilon_y'' &= \nu(\lambda - 1)\ d\sigma_x \\ E_o\, d\gamma_{xy}'' &= (\lambda - 1)(1 + \nu)\ d\tau_{xy} \end{aligned}\right\} \qquad (28)$$

These plastic increments of strain vanish if

$$\left.\begin{aligned} d\sigma_x &= 0 \\ d\tau_{xy} &= 0 \end{aligned}\right\} \qquad (29)$$

According to Kaufmann's stress-strain relation, neutral changes of stress are characterized by the two conditions given as equations (29). If the most general change of stress considered here is represented by a point with the coordinates $d\sigma_x$, $d\sigma_y$, and $d\tau_{xy}$ in a three-dimensional space, the condition of equation (21) represents a plane through the origin which separates the "region of loading" from the "region of unloading." Equations (29), however, define a straight line which does not mark off two such regions.

Ilyushin (reference 6) considers an incompressible material and assumes the stress-strain relations for loading to have the form

$$\left.\begin{aligned} E_o \epsilon_x &= \frac{2\sigma_x - \sigma_y}{2(1 - \omega)} \\ E_o \epsilon_y &= \frac{2\sigma_y - \sigma_x}{2(1 - \omega)} \\ E_o \gamma_{xy} &= \frac{3\tau_{xy}}{1 - \omega} \end{aligned}\right\} \qquad (30)$$

where $\omega$ is a function of the intensity of stress $\sigma_i$ defined by

$$\sigma_i = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2} \tag{31}$$

For loading, the increments of stress and strain are then connected by

$$\left.\begin{aligned}
E_o\, d\epsilon_x &= \frac{1}{2(1 - \omega)}\left(2d\sigma_x - d\sigma_y - \frac{2\sigma_x - \sigma_y}{1 - \omega}\, d\omega\right)\\
E_o\, d\epsilon_y &= \frac{1}{2(1 - \omega)}\left(2d\sigma_y - d\sigma_x - \frac{2\sigma_y - \sigma_x}{1 - \omega}\, d\omega\right)\\
E_o\, d\gamma_{xy} &= \frac{3}{1 - \omega}\left(d\tau_{xy} - \frac{\tau_{xy}}{1 - \omega}\, d\omega\right)
\end{aligned}\right\} \tag{32}$$

For buckling from a state of uniform compression $\sigma_x = -\sigma_o$, in particular,

$$d\omega = \frac{d\omega}{d\sigma_i}\, d\sigma_i = -\frac{\omega'}{2}\left(2d\sigma_x - d\sigma_y\right) \tag{33}$$

where

$$\omega' = d\omega/d\sigma_i \tag{34}$$

Equations (32) then reduce to

$$\left.\begin{aligned}
E_o\, d\epsilon_x &= \frac{1}{2(1 - \omega)^2}\left(1 - \omega - \omega'\sigma_o\right)\left(2d\sigma_x - d\sigma_y\right)\\
E_o\, d\epsilon_y &= -\frac{1}{2(1 - \omega)^2}\left\{\left(1 - \omega + 2\sigma_o\omega'\right) d\sigma_x + \left[2(1 - \omega) - \frac{\sigma_o\omega'}{2}\right] d\sigma_y\right\}\\
E_o\, d\gamma_{xy} &= \frac{3}{1 - a}\, d\tau_{xy}
\end{aligned}\right\} \tag{35}$$

For unloading, the relations, equations (2), are supposed to hold with $\nu = 1/2$ on account of the assumed incompressibility of the plate material:

$$\left.\begin{aligned} E_o\, d\epsilon_x &= \frac{1}{2}\left(2d\sigma_x - d\sigma_y\right) \\ E_o\, d\epsilon_y &= \frac{1}{2}\left(2d\sigma_y - d\sigma_x\right) \\ E_o\, d\gamma_{xy} &= 3d\tau_{xy} \end{aligned}\right\} \tag{36}$$

As to the criterion for loading and unloading, this is again supposed to be given by the sign of the expression

$$dW = \sigma_x\, d\epsilon_x + \sigma_y\, d\epsilon_y + \tau_{xy}\, d\gamma_{xy} \tag{37}$$

In particular, it is given by the sign of

$$-\sigma_o\, d\epsilon_x \tag{38}$$

in the case of buckling from a state of uniform compression $\sigma_x = -\sigma_o$. In view of the first of equations (35), this means that neutral changes of stress are again characterized by equation (21). It is easily seen, however, that for $2d\sigma_x - d\sigma_y = 0$, equations (35) and (36) do not give the same increments of strain. Ilyushin's stress-strain relations are thus seen to exhibit an objectionable discontinuity along the surface $2d\sigma_x - d\sigma_y = 0$ which separates the region of loading from the region of unloading. A more detailed analysis of the effect of this discontinuity in the case of a buckling plate is found in appendix A.

## Fundamental Equation of Plastic Buckling of a Simply Compressed Plate

The technique used in the derivation of the fundamental equation of the plastic buckling of a simply compressed plate is quite similar to that needed for the same problem in the elastic range. (See, for example, the more general problem of combined bending and compression of elastic plates in reference 1, p. 302.) There is one essential difference, however, in that the stress-strain relations given in equations (26) must be used in the regions of loading rather than generalized Hooke's law. Consequently, the middle plane of the unbent plate will no longer play the role of the neutral surface in the buckled position. Once the position of the neutral surface has been found and the bending and twisting moments determined as functions of the second derivatives of the deflection of the plate, the equilibrium conditions and the final differential equation can be derived in exactly the same fashion as that used by Timoshenko in reference 1.

It will be found more convenient, in the following discussion, to use "reduced stresses" rather than actual stresses, that is, stresses reduced by dividing the actual stress by Young's modulus $E_o$. No new notation will be employed to denote these reduced stresses; therefore, care must be taken in interpreting the results obtained here in terms of the known facts for elastic buckling. An attempt will be made at such points to keep the notation clear. In addition, the use of differentials of stress and strain may lead to some confusion in deriving the equations of equilibrium for an element. Since the stress-strain relations given in equations (4) and (26) are linear in these differentials, both sides of the equations may be divided by $dt > 0$, where $t$ may be regarded as the time. It should be noted that $t$ appears homogeneously; that is, the time scale may be arbitrarily distorted without changing the equations. If differentiation with respect to $t$ is denoted by a dot, equations (4) and (26) can be rewritten as reduced stress-strain relations, for $\dot{W} \geqq 0$,

$$\left.\begin{aligned}
\dot{\epsilon}_x &= \lambda\dot{\sigma}_x - \left(\nu + \frac{\lambda - 1}{2}\right)\dot{\sigma}_y \\
\dot{\epsilon}_y &= -\left(\nu + \frac{\lambda - 1}{2}\right)\dot{\sigma}_x + \frac{\lambda + 3}{4}\dot{\sigma}_y \\
\dot{\epsilon}_z &= -\left(\nu + \frac{\lambda - 1}{2}\right)\dot{\sigma}_x - \left(\nu - \frac{\lambda - 1}{4}\dot{\sigma}_y\right) \\
\dot{\gamma}_{xy} &= 2(1 + \nu)\dot{\tau}_{xy}
\end{aligned}\right\} \tag{39}$$

and for $\dot{W} \leqq 0$,

$$\left.\begin{aligned} \dot{\epsilon}_x &= \dot{\sigma}_x - \nu\dot{\sigma}_y \\ \dot{\epsilon}_y &= -\nu\dot{\sigma}_x + \dot{\sigma}_y \\ \dot{\epsilon}_z &= -\nu\dot{\sigma}_x - \nu\dot{\sigma}_y \\ \dot{\gamma}_{xy} &= 2(1 + \nu)\dot{\tau}_{xy} \end{aligned}\right\} \tag{40}$$

where $\dot{W} = dW/dt$.

The stress rates $\dot{\sigma}_x$ and $\dot{\sigma}_y$ can be found in terms of the corresponding strain rates for loading by solving the first two of equations (39). Thus,

$$\left.\begin{aligned} \dot{\sigma}_x &= \frac{1}{(5 - 4\nu)\lambda - (1 - 2\nu)^2}\left[(\lambda + 3)\dot{\epsilon}_x + 2(\lambda - 1 + 2\nu)\dot{\epsilon}_y\right] \\ \dot{\sigma}_y &= \frac{1}{(5 - 4\nu)\lambda - (1 - 2\nu)^2}\left[2(\lambda - 1 + 2\nu)\dot{\epsilon}_x + 4\lambda\dot{\epsilon}_y\right] \end{aligned}\right\} \tag{41}$$

The criterion for loading

$$2\dot{\sigma}_x - \dot{\sigma}_y < 0 \tag{42}$$

can then be written as

$$\frac{1}{(5 - 4\nu)\lambda - (1 - 2\nu)^2}\left[(2 - \nu)\dot{\epsilon}_x + (2\nu - 1)\dot{\epsilon}_y\right] < 0 \tag{43}$$

Now, Poisson's ratio $\nu$ satisfies the inequality $-1 \leqq \nu \leqq \frac{1}{2}$ (cf. reference 13, p. 104); in addition, $\lambda \geqq 1$. Consequently, the expression appearing outside the brackets is always positive and the inequality, equation (43), can be replaced by

$$(2 - \nu)\dot{\epsilon}_x + (2\nu - 1)\dot{\epsilon}_y < 0 \tag{44}$$

The strain rates appearing in equations (39) and (40) must now be evaluated. The strain rates in the middle surface will be denoted by $\dot{\epsilon}_1 = \dot{\epsilon}_1(x,y)$, the normal strain rate in the x-direction; $\dot{\epsilon}_2 = \dot{\epsilon}_2(x,y)$, the normal strain rate in the y-direction; and $\dot{\gamma} = \dot{\gamma}(x,y)$, the rate of shear strain. Points on the normal to the undeformed middle surface are assumed to remain on the normal of the bent middle surface. This implies that the strain rates $\dot{\epsilon}_x$, $\dot{\epsilon}_y$, and $\dot{\gamma}_{xy}$ at any point of the plate can be written in the following form:

$$\left.\begin{aligned} \dot{\epsilon}_x &= \dot{\epsilon}_1 - z\dot{K}_1 \\ \dot{\epsilon}_y &= \dot{\epsilon}_2 - z\dot{K}_2 \\ \dot{\gamma}_{xy} &= \dot{\gamma} - z\dot{K}_{12} \end{aligned}\right\} \tag{45}$$

The quantities $\dot{K}_1$, $\dot{K}_2$, and $\dot{K}_{12}$ appearing in equations (45) are defined in terms of the rate of deflection $\dot{w} = \dot{w}(x,y)$ of the middle surface in the following way:

$$\left.\begin{aligned} \dot{K}_1 &= \frac{\partial^2 \dot{w}}{\partial x^2} \\ \dot{K}_2 &= \frac{\partial^2 \dot{w}}{\partial y^2} \\ \dot{K}_{12} &= 2\,\frac{\partial^2 \dot{w}}{\partial x \partial y} \end{aligned}\right\} \tag{46}$$

Geometrically, $\dot{K}_1$ and $\dot{K}_2$ represent the rates of curvature of the middle surface in the x- and y-directions, respectively, whereas $\dot{K}_{12}$ represents the rate of relative twist. The criterion for loading, equation (44), can now be rewritten in terms of the strains of the middle surface and the quantities $\dot{K}_1$, $\dot{K}_2$, and $\dot{K}_{12}$. It is seen that loading takes place provided

$$(2 - \nu)\dot{\epsilon}_1 + (2\nu - 1)\dot{\epsilon}_2 < z\left[(2 - \nu)\dot{K}_1 + (2\nu - 1)\dot{K}_2\right] \tag{47}$$

With the word "sign" to denote the sign of the quantity within the parenthesis and vertical bars to denote the absolute value of the enclosed expression, the last inequality may be transformed into

$$z \text{ sign } \left[(2 - \nu)\dot{K}_1 + (2\nu - 1)\dot{K}_2\right] > \frac{(2 - \nu)\dot{\epsilon}_1 + (2\nu - 1)\dot{\epsilon}_2}{\left|(2 - \nu)\dot{K}_1 + (2\nu - 1)\dot{K}_2\right|} \tag{48}$$

This inequality can be simplified further by introducing two new quantities $\dot{K}$ and $z_o$ defined by

$$\left.\begin{aligned} \dot{K} &= (2 - \nu)\dot{K}_1 + (2\nu - 1)\dot{K}_2 \\ z_o &= \frac{(2 - \nu)\dot{\epsilon}_1 + (2\nu - 1)\dot{\epsilon}_2}{\dot{K}} \end{aligned}\right\} \tag{49}$$

The inequality, equation (48), becomes then

$$z \text{ sign } (\dot{K}) > \dot{K}z_o/|\dot{K}| = z_o \text{ sign } (\dot{K}) \tag{50}$$

that is,

$$\left.\begin{array}{l} z > z_o \text{ for positive } \dot{K} \\ \\ z < z_o \text{ for negative } \dot{K} \end{array}\right\} \qquad (51)$$

The surface $z = z_o$ separates the regions of loading and unloading in the plate; a given part is in a state of loading or not according to which condition of equations (51) is satisfied.

The criterion just developed must now be applied to the problem of buckling. As mentioned previously, the stress distribution of the buckled plate differs from the original state of pure compression by certain additional stresses $\dot{\sigma}_x\,dt$, $\dot{\sigma}_y\,dt$, and $\dot{\tau}_{xy}\,dt$. These new stresses are such that their total stress resultants must vanish and the moments produced will be in equilibrium with the moment generated by the original compressive force in the buckled plate. The vanishing of the stress resultants will lead to a formula for $z_o$ in terms of the constants of the material and the value of $\sigma_o$. Once this equation has been developed, a rather straightforward computation will lead to the desired equation of equilibrium.

The rates $\dot{N}_x$ and $\dot{N}_y$ of the stress resultants are defined as

$$\left.\begin{array}{l} \dot{N}_x = \displaystyle\int_{-h/2}^{h/2} \dot{\sigma}_x\,dz \\ \\ \dot{N}_y = \displaystyle\int_{-h/2}^{h/2} \dot{\sigma}_y\,dz \end{array}\right\} \qquad (52)$$

As indicated in equations (51), two cases must be considered according to whether $\dot{K} > 0$ or $\dot{K} < 0$. For $\dot{K} > 0$, direct computation shows that

$$\left.\begin{aligned} \dot{N}_x - \nu\dot{N}_y &= \dot{\epsilon}_1 h + c\dot{K}\left(\frac{h}{2} - z_o\right)^2 \\ \dot{N}_y - \nu\dot{N}_x &= \dot{\epsilon}_2 h - \frac{1}{2}c\dot{K}\left(\frac{h}{2} - z_o\right)^2 \end{aligned}\right\} \tag{53}$$

where the quantity $c$ is a function of $\lambda$ and $\nu$ given by

$$c = \frac{\lambda - 1}{(5 - 4\nu)\lambda - (1 - 2\nu)^2} \tag{54}$$

Appendix B contains the details of this calculation and others used in this section. It has been pointed out previously that $\dot{N}_x$ and $\dot{N}_y$ must vanish. According to equations (53), this yields

$$\left(\frac{h}{2} - z_o\right)^2 = -\frac{\dot{\epsilon}_1 h}{c\dot{K}} = \frac{2\dot{\epsilon}_2 h}{c\dot{K}} \tag{55}$$

Thus the strain rates $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ in the middle surface are related by the equation

$$\dot{\epsilon}_1 = -2\dot{\epsilon}_2 \tag{56}$$

From the definition of $z_o$, equations (49), and this result, it is seen that

$$\begin{aligned} z_o &= (2 - \nu)\frac{\dot{\epsilon}_1}{\dot{K}} + (2\nu - 1)\frac{\dot{\epsilon}_2}{\dot{K}} \\ &= -\frac{\dot{\epsilon}_2}{\dot{K}}(5 - 4\nu) \end{aligned} \tag{57}$$

This result may then be substituted back into equation (56) to yield

$$\left(\frac{h}{2} - z_o\right)^2 = \frac{2\dot{\epsilon}_2 h}{c\dot{K}} = - \frac{2z_o h}{c(5 - 4\nu)} \tag{58}$$

Relation (58) can be solved for $z_o$ to yield the equation of the neutral surface which separates the regions of loading and unloading. Since $h$, $c$, and $\nu$ depend only on the geometry of the plate and the applied compressive stress, $z_o$ will depend only on these quantities. It is more convenient to introduce a new quantity $\zeta_o$ defined by

$$\zeta_o = \frac{2z_o}{h} \tag{59}$$

Then the quadratic equation for $z_o$, equation (58), becomes a quadratic equation in $\zeta_o$, namely,

$$\left(1 - \zeta_o\right)^2 + \frac{4\zeta_o}{c(5 - 4\nu)} = 0 \tag{60}$$

There are two solutions to this equation, in general, but the only one which is physically realizable is

$$\zeta_o = \zeta_o^+ = \alpha + \sqrt{\alpha^2 - 1} \tag{61}$$

where

$$\alpha = 1 - \frac{2}{c(5 - 4\nu)} \tag{62}$$

Equation (61) gives the desired formula for the neutral surface.

When $\dot{K} < 0$, the procedure is exactly the same as that outlined. Again the details are found in appendix B. The formula for the neutral surface is given in this case by

$$\zeta_o = \zeta_o^- = -\alpha - \sqrt{\alpha^2 - 1} \tag{63}$$

Roughly speaking, the sign of $\dot{K}$ indicates whether the plate "buckles up" or "buckles down." Consequently, the differences in sign found by comparing equations (61) and (63) are quite natural. It would also be expected that the rates of change of the bending and twisting moments, as well as the resulting equilibrium equation, should be independent of the sign of $\dot{K}$. This will be shown to be true.

The rates of change of the bending and twisting moments can be computed now that $z_o$ or $\zeta_o$ is known for $\dot{K} > 0$ and for $\dot{K} < 0$. The rates of change of the bending moments, $\dot{M}_x$ and $\dot{M}_y$, are defined as

$$\left.\begin{aligned} \dot{M}_x &= \int_{-h/2}^{h/2} \dot{\sigma}_x z \, dz \\ \dot{M}_y &= \int_{-h/2}^{h/2} \dot{\sigma}_y z \, dz \end{aligned}\right\} \tag{64}$$

where the moments are taken about the y- and x-axes, respectively. The rate of change of the twisting moment $\dot{M}_{xy}$ is given by

$$\dot{M}_{xy} = -\int_{-h/2}^{h/2} \tau_{xy} z \, dz \tag{65}$$

The calculation of the rates, equations (64) and (65), must be carried out separately for $\dot{K} > 0$ and $\dot{K} < 0$. It can be shown that the only quantity appearing in the final result which depends on the sign of $\dot{K}$ is the function $\delta$ defined by

$$\left.\begin{aligned} \delta &= \frac{1}{2}\left[1 - \frac{3}{2}\zeta_o^+ + \frac{1}{2}\left(\zeta_o^+\right)^3\right] \qquad \text{for } \dot{K} > 0 \\ \delta &= \frac{1}{2}\left[1 + \frac{3}{2}\zeta_o^- - \frac{1}{2}\left(\zeta_o^-\right)^3\right] \qquad \text{for } \dot{K} < 0 \end{aligned}\right\} \tag{66}$$

According to equation (63), $\zeta_o^- = -\zeta_o^+$; the numerical value of $\delta$ obtained from equations (66) will therefore be the same in either case. Thus the expressions for $\dot{M}_x$, $\dot{M}_y$, and $\dot{M}_{xy}$ will be the same in both cases. The details are found in appendix B in which it is shown that

$$\dot{M}_x = -\frac{h^3}{12(1 - \nu^2)}\left\{\dot{K}_1\left[1 - c\delta(2 - \nu)^2\right] + \dot{K}_2\left[\nu - c\delta(2 - \nu)(2\nu - 1)\right]\right\} \tag{67}$$

$$\dot{M}_y = -\frac{h^3}{12(1 - \nu^2)}\left\{\dot{K}_1\left[\nu - c\delta(2 - \nu)(2\nu - 1)\right] + \dot{K}_2\left[1 - c\delta(2\nu - 1)^2\right]\right\} \tag{68}$$

$$\dot{M}_{xy} = \frac{h^3}{12(1 - \nu^2)}\left[\frac{(1 - \nu)}{2}\dot{K}_{12}\right] \tag{69}$$

The equation of equilibrium can be set up in terms of the bending and twisting moments and compressive load without reference to the stress-strain relations. This has already been done by Timoshenko (reference 1, p. 305) for the more general case of combined bending and tension or compression. His results may be applied to this special case of a simply compressed plate. With the present notation, the equation of equilibrium is

$$\frac{\partial^2 \dot{M}_x}{\partial x^2} - 2\frac{\partial^2 \dot{M}_{xy}}{\partial x \partial y} + \frac{\partial^2 \dot{M}_y}{\partial y^2} = -N_x \frac{\partial^2 \dot{w}}{\partial x^2} \tag{70}$$

Timoshenko's relation was originally written in terms of the actual bending moments and actual compressive stress resultant $N_x$ rather than the rates of the reduced quantities. Timoshenko's equation can be differentiated with respect to time and divided by $E_o$ on both sides, so that equation (70) is the desired equation of equilibrium provided $N_x$ is defined as

$$N_x = \sigma_o h / E_o \tag{71}$$

From equations (46),

$$\dot{K}_1 = \partial^2 \dot{w} / \partial x^2$$

$$\dot{K}_2 = \partial^2 \dot{w} / \partial y^2$$

$$\dot{K}_{12} = 2\partial^2 \dot{w} / \partial x \partial y$$

With these relations and equations (67), (68), and (69), equation (70) may be rewritten as

$$D_{11}\dot{w}_{xxxx} + 2D_{12}\dot{w}_{xxyy} + D_{22}\dot{w}_{yyyy} = -\sigma_o h \dot{w}_{xx} \tag{72}$$

where the subscripts denote partial differentiation with respect to the variable named and

$$\left.\begin{aligned} D_{11} &= D\left[1 - c\delta(2 - \nu)^2\right] \\ D_{12} &= D\left[1 - c\delta(2 - \nu)(2\nu - 1)\right] \\ D_{22} &= D\left[1 - c\delta(2\nu - 1)^2\right] \\ D &= \frac{h^3E_o}{12(1 - \nu^2)} \end{aligned}\right\} \qquad (73)$$

The quantity $D$ is the well-known flexural rigidity of the plate. Equation (72) resembles the equation for the buckling of an elastic anisotropic plate (reference 1, p. 380). There is one important difference, however. In the case of an elastic anisotropic plate, the coefficients $D_{11}$, $D_{12}$, and $D_{22}$ are constants of the material; for the plastic case $D_{11}$, $D_{12}$, and $D_{22}$ are functions of $\sigma_o$. In other words, the plate is anisotropic but this anisotropy is caused by and is a function of the compressive stress. Consequently, certain changes must be made in the standard procedure for calculating buckling loads for anisotropic plates. Several examples illustrating this technique are given in the succeeding sections. Graphs of the quantities

$$D_{11}' = 1 - c\delta(2 - \nu)^2$$

$$D_{12}' = 1 - c\delta(2 - \nu)(2\nu - 1)$$

and

$$D_{22}' = 1 - c\delta(2\nu - 1)^2$$

as functions of $\lambda$ for $\nu = 0.32$ are given in figure 3 and the numerical values are listed in table I.

## Typical Boundary Conditions for the Fundamental Equation for a Simply Compressed Plate

The discussion of the boundary conditions for the buckling equation, equation (72), is facilitated by expressing the moment rates $\dot{M}_x$, $\dot{M}_y$, and $\dot{M}_{xy}$ in terms of the second derivatives of the deflection rate $\dot{w}$ and the stiffnesses $D_{11}$, $D_{12}$, and $D_{22}$ introduced in equations (73). Thus, it follows from equations (67), (68), and (69) that

$$\left.\begin{aligned} E_o\dot{M}_x &= -D_{11}\dot{w}_{xx} - \left[D_{12} - (1 - \nu)D\right]\dot{w}_{yy} \\ E_o\dot{M}_y &= -\left[D_{12} - (1 - \nu)D\right]\dot{w}_{xx} - D_{22}\dot{w}_{yy} \\ E_o\dot{M}_{xy} &= D(1 - \nu)\dot{w}_{xy} \end{aligned}\right\} \tag{74}$$

The following boundary conditions are typical in the buckling of rectangular plates:

(1) Simply supported edge at $x = 0$.- The deflection rate $\dot{w}$ and the moment rate $\dot{M}_x$ must vanish at this edge; that is,

$$\left.\begin{aligned} &\dot{w} = 0 \\ &-D_{11}\dot{w}_{xx} - \left[D_{12} - (1 - \nu)D\right]\dot{w}_{yy} = 0 \end{aligned}\right\} \tag{75}$$

for $x = 0$.

(2) Built-in edge at $x = 0$.- At this edge, the deflection rate $\dot{w}$ and the slope rate $\dot{w}_x$ must vanish; that is,

$$\left.\begin{aligned} \dot{w} &= 0 \\ \dot{w}_x &= 0 \end{aligned}\right\} \tag{76}$$

for $x = 0$.

(3) Free edge at $y = 0$.- For a free edge, the rates of the bending moment $\dot{M}_y$ and of the equivalent shear load $\left(-2\partial\dot{M}_{xy}/\partial x\right) + \left(\partial\dot{M}_y/\partial y\right)$ must vanish (reference 1, p. 300). Consequently,

$$\left.\begin{aligned} \left[D_{12} - (1 - \nu)D\right]\dot{w}_{xx} + D_{22}\dot{w}_{yy} &= 0 \\ \left[D_{12} + (1 - \nu)D\right]\dot{w}_{xxy} + D_{22}\dot{w}_{yyy} &= 0 \end{aligned}\right\} \qquad (77)$$

for $y = 0$.

(4) Plane of symmetry at $y = 0$.- If the buckled shape of the plate is symmetrical with respect to the plane $y = 0$, the rates of the slope $\dot{w}_y$ and of the equivalent shear load $\left(-2\partial\dot{M}_{xy}/\partial x\right) + \left(\partial\dot{M}_y/\partial y\right)$ will vanish. Therefore,

$$\left.\begin{aligned} \dot{w}_y &= 0 \\ \left[D_{12} + (1 - \nu)D\right]\dot{w}_{xxy} + D_{22}\dot{w}_{yyy} &= 0 \end{aligned}\right\} \qquad (78)$$

or

$$\left.\begin{aligned} \dot{w}_y &= 0 \\ \dot{w}_{yyy} &= 0 \end{aligned}\right\} \qquad (79)$$

for $y = 0$. Should these boundary conditions be given on other edges, the necessary changes in the formulas can be made easily.

Several examples of the buckling of a simply compressed plate with various boundary conditions of the type just discussed are considered in the next section. In all these examples it is assumed that the plate is in the state of compression previously described, $\sigma_x = -\sigma_o$, and that the edges $x = a$ and $x = 0$, perpendicular to the direction of the compressive force, are simply supported. The other boundary conditions are specified for each example. Equations (75) are satisfied at $x = 0$ and $x = a$ if the deflection rate is written in the form

$$\dot{w} = f(y) \sin (m\pi x/a) \tag{80}$$

where $m$ is an integer. Thus a section of the plate in the buckled state, obtained by setting $y$ = Constant, is described by a series of sine waves; the integer $m$ gives the number of half waves. Substitution of this expression for $\dot{w}$ into the partial differential equation (72) yields the ordinary differential equation

$$(m\pi/a)^4 D_{11} f - 2(m\pi/a)^2 D_{12} f^{ii} + D_{22} f^{iv} = \sigma_o h (m\pi/a)^2 f \tag{81}$$

in which the Roman numerals denote the corresponding derivatives with respect to $y$. With

$$\left.\begin{aligned} p^2 &= \left(\frac{m\pi}{a}\right)^2 \frac{D_{12}}{D_{22}} \\ q^2 &= \frac{\sigma_o h}{D_{12}} - \left(\frac{m\pi}{a}\right)^2 \frac{D_{11}D_{22} - D_{12}^2}{D_{12}D_{22}} \end{aligned}\right\} \tag{82}$$

equations (82) can be written in the form

$$f^{iv} - 2p^2 f^{ii} + p^2\left(p^2 - q^2\right) f = 0 \tag{83}$$

The general solution of this equation in $f(y)$ is

$$f(y) = c_1 \cosh ry + c_2 \sinh ry + c_3 \cos sy + c_4 \sin sy \tag{84}$$

where $c_1$, $c_2$, $c_3$, and $c_4$ are arbitrary constants which must be determined from the boundary conditions, and

$$\left.\begin{aligned} r &= \sqrt{p(q + p)} \\ s &= \sqrt{p(q - p)} \end{aligned}\right\} \tag{85}$$

Equation (84) is the fundamental relation which must be studied for each particular case of buckling from a state of simple compression. The boundary conditions of the type discussed lead to linear homogeneous equations for the constants $c_1$, $c_2$, $c_3$, and $c_4$. The condition that these equations possess solutions $c_1$, $c_2$, $c_3$, and $c_4$ which are not all zero yields an equation in $r$ and $s$ from which the critical stress $\sigma_o$ can be determined.

## Specific Examples

Plastic buckling of a narrow strip; relation of present theory to beam theory.- It seems worth while to investigate how the present theory of the plastic buckling of plates is related to the Engesser-Von Kármán theory of the plastic buckling of beams. Consider a rectangular plate of thickness $h$ which is simply supported along $x = 0$ and $x = a$, free along $y = \pm b/2$, and under the compressive stress $\sigma_x = -\sigma_o$. As $b \longrightarrow 0$ for a fixed value of $a$, the buckling condition for this plate might be expected to approach that of a beam which has the length $a$, is simply supported at its two ends, and possesses a rectangular cross section of height $h$ and width $b$.

The solution given by equation (80) automatically fulfills the boundary conditions at $x = 0$ and $x = a$. In addition, the symmetry condition along $y = 0$, equations (79), requires that the coefficients in equation (84) satisfy

$$rc_2 + sc_4 = 0 \tag{86}$$

and

$$r^3c_2 - s^3c_4 = 0 \tag{87}$$

Accordingly,

$$c_2 = c_4 = 0 \tag{88}$$

if the function $f(y)$ is not to vanish identically. Boundary conditions, equations (77), for a free edge along $y = b/2$ furnish the equations

$$\left.\begin{aligned}
&-\frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\left(c_1 \cosh\frac{rb}{2} + c_3 \cos\frac{sb}{2}\right) \\
&\qquad + D_{22}\left(r^2c_1 \cosh\frac{rb}{2} - s^2c_3 \cos\frac{sb}{2}\right) = 0 \\
&-\frac{m^2\pi^2}{a^2}\left[D_{12} + (1 - \nu)D\right]\left(rc_1 \sinh\frac{rb}{2} - sc_3 \sin\frac{sb}{2}\right) \\
&\qquad + D_{22}\left(r^3c_1 \sinh\frac{rb}{2} + s^3c_3 \sin\frac{sb}{2}\right) = 0
\end{aligned}\right\} \tag{89}$$

The boundary conditions corresponding to a free edge at $y = -b/2$ are fulfilled automatically if equations (89) are satisfied. Now these relations are linear homogeneous equations in $c_1$ and $c_3$; if they are to have nonvanishing solutions, the determinant of the coefficients must be zero. Therefore,

$$\begin{vmatrix} \left\{D_{22}r^2 - \frac{m^2\pi^2}{a^2}\left[D_{12} - (1-\nu)D\right]\right\}\cosh\frac{rb}{2} & -\left\{D_{22}s^2 + \frac{m^2\pi^2}{a^2}\left[D_{12} - (1-\nu)D\right]\right\}\cos\frac{sb}{2} \\ r\left\{D_{22}r^2 - \frac{m^2\pi^2}{a^2}\left[D_{12} + (1-\nu)D\right]\right\}\sinh\frac{rb}{2} & s\left\{D_{22}s^2 + \frac{m^2\pi^2}{a^2}\left[D_{12} + (1-\nu)D\right]\right\}\sin\frac{sb}{2} \end{vmatrix} = 0 \tag{90}$$

This determinantal equation can be reduced to the form

$$\left(\xi^2 - m^2\pi^2R^2Q\right)^2\eta\tan\frac{\eta}{2} + \left(\eta^2 + m^2\pi^2R^2Q\right)^2\xi\tanh\frac{\xi}{2} = 0 \tag{91}$$

where

$$\left.\begin{aligned} R &= b/a \\ \xi &= rb = b\sqrt{p(q + p)} \\ \eta &= sb = b\sqrt{p(q - p)} \\ Q &= \left[D_{12} - (1 - \nu)D\right]\Big/D_{22} \end{aligned}\right\} \tag{92}$$

If, for a fixed value of the span $a$, the width $b$ of the plate approaches zero, $R$ and hence $\xi$ and $\eta$ tend toward zero too. Accordingly, the functions $\xi \tanh(\xi/2)$ and $\eta \tan(\eta/2)$ appearing in equation (91) may be replaced by $\xi^2/2$ and $\eta^2/2$, respectively. For $m = 1$, in particular, this relation can be reduced to

$$kD_{22}' - \Delta D_{22}'^2 - D_{12}'^2 + Q^2 D_{22}'^2 = 0 \tag{93}$$

in which the following notation has been used

$$\left.\begin{aligned} k &= 12\frac{\sigma_o a^2}{\pi^2 h^2 E_o}\left(1 - \nu^2\right) \\ D_{11}' &= D_{11}/D \\ D_{12}' &= D_{12}/D \\ D_{22}' &= D_{22}/D \\ \Delta &= \left(D_{11}'D_{22}' - D_{12}'^2\right)\Big/D_{22}'^2 \end{aligned}\right\} \tag{94}$$

A detailed development of equation (93) is presented in appendix C. If the plate under consideration buckles within the elastic range, $D_{11}' = D_{12}' = D_{22}' = 1$; consequently, $\Delta = 0$ and $Q = \nu$. According to equation (93) then, $k = 1 - \nu^2$ and

$$\sigma_o = \frac{\pi^2 h^2 E_o}{12a^2} \tag{95}$$

The compressive force $P = \sigma_o bh$ under which the plate buckles is thus seen to equal

$$P = \frac{\pi^2 h^3 b E_o}{12a^2} = \frac{\pi^2 EI}{a^2} \tag{96}$$

where $I = bh^3/12$ is the cross-sectional moment of inertia. Equation (96) is Euler's formula for the elastic buckling of a simply supported beam of span a.

If the plate buckles after the compressive stress $\sigma_o$ has exceeded the limit of proportionality, the evaluation of equation (93) becomes more difficult. Noting the relations given in equations (94), equation (93) can be transformed into

$$k = D_{11}' - Q^2 D_{22}' \tag{97}$$

For $\nu = \frac{1}{2}$, this relation can be handled quite simply. A straightforward computation (see appendix C) yields the following result for the critical compressive stress $\sigma_{cr} = \sigma_o$:

$$\sigma_{cr} = \frac{\pi^2 E_o h^2}{3a^2} \frac{1}{\left(1 + \sqrt{\lambda}\right)^2} \tag{98}$$

The critical buckling load P is given by

$$P = \sigma_{cr} bh = \frac{\pi^2 E_o I}{a^2} \frac{4}{\left(1 + \sqrt{\lambda}\right)^2} \tag{99}$$

where again $I = bh^3/12$ is the cross-sectional moment of inertia. Hence the reduced modulus $E^*$ is given by

$$E^* = \frac{4E_o}{\left(1 + \sqrt{\lambda}\right)^2} = \frac{4E_oE}{\left(\sqrt{E} + \sqrt{E_o}\right)^2} \tag{100}$$

according to the definition of $\lambda$ given by equation (13). This reduced modulus is identical with that found in the Engesser-Von Kármán theory of the buckling of beams beyond the elastic limit. (See references 2 and 3.) Thus in the case of an incompressible material, $\nu = \frac{1}{2}$, the critical load for a beam can be found as a limiting case from the theory of plates. It should be noted that, within the framework of beam theory, the critical stress is independent of the value of Poisson's ratio $\nu$.

This result is not necessarily true for materials which are not incompressible, $\nu \neq \frac{1}{2}$. The quantity

$$\frac{k}{1 - \nu^2} = \frac{12\sigma_o a^2}{\pi^2 h^2 E_o} \tag{101}$$

which is simply a constant multiple of the critical stress for a given plate, has been evaluated as a function of $\lambda$ for the case just mentioned, $\nu = \frac{1}{2}$, and for a material with $\nu = 0.32$; the results are plotted in figure 4. Although the two functions agree at $\lambda = 1$, as previously proved, there is a marked difference which increases with increasing values of $\lambda$. Consequently, two beams of the same shape but with different values of Poisson's ratio will buckle at different critical stresses.

At first sight, these results may seem somewhat contradictory since the analysis of plates developed herein is a generalization of the Engesser-Von Kármán theory of the buckling of beams and yet this theory does not always appear as a limiting case of the present analysis. An explanation of this discrepancy is afforded by a closer examination of the position of the neutral surface $z_o$ as a function of the parameter $\lambda$. The function $z_o$ depends on the plate thickness $h$ and on $\nu$ and $\lambda$. The limiting process considered does not affect this relation and the neutral line in the beam is determined by the intersection of the neutral surface in the plate and the vertical plane $y = \text{Constant}$. The neutral line so determined need not coincide with the neutral line determined directly from the theory of beams. This difference is the basis of the discrepancies in $k/(1 - \nu^2)$ noted.

According to equations (49), the function $z_o$ is given by

$$z_o = \frac{(2 - \nu)\dot{\epsilon}_1 + (2\nu - 1)\dot{\epsilon}_2}{\dot{K}}$$

where

$$\dot{K} = (2 - \nu)\dot{K}_1 + (2\nu - 1)\dot{K}_2$$

The quantities $\dot{\epsilon}_1$ and $\dot{\epsilon}_2$ are the strain rates in the middle surface in the x- and y-directions, respectively. For the small displacements studied herein, $\dot{K}_1$ is the rate of curvature of the line of intersection of the plane $y = \text{Constant}$ and the middle surface. Similarly, $\dot{K}_2$ is the rate of curvature of the line of intersection of the plane $x = \text{Constant}$ and the middle surface. In general, the position of the neutral surface depends on the four quantities $\dot{\epsilon}_1$, $\dot{\epsilon}_2$, $\dot{K}_1$, and $\dot{K}_2$ and this holds in the transition from plate to beam. On the other hand, the Engesser-Von Kármán theory of beams assumes that the position of the neutral line depends on $\dot{\epsilon}_1$ and $\dot{K}_1$ alone. (See reference 1, p. 158.) The position of the neutral line will depend on the type of analysis used. Specifically, it can be shown (see appendix C) that the position of the neutral line found by considering the intersection of the neutral surface and a plane $y = \text{Constant}$ and the position determined by the Engesser-Von Kármán method agree if and only if $\nu = \frac{1}{2}$. Therefore, it is to be expected that the transition from plate to beam is valid only under this condition.

Buckling of a simply supported plate.- The critical load for a simply supported plate under edge thrusts, stressed beyond the elastic limit, has been discussed by Timoshenko (reference 1, p. 387) on a purely formal basis. The object of this section is the development of an analytical formula for the critical stress of such a plate on the basis of equation (72) and the subsequent comparison with Timoshenko's results.

According to equations (75), the boundary conditions for a plate simply supported at all four sides can be written in the form

$$
\left.
\begin{aligned}
&\dot{w} = 0 \\
&-D_{11}\dot{w}_{xx} - \left[D_{12} - (1 - \nu)D\right]\dot{w}_{yy} = 0
\end{aligned}
\right\} \qquad (102)
$$

at $x = 0$ and $x = a$, and

$$
\left.
\begin{aligned}
&\dot{w} = 0 \\
&-D_{22}\dot{w}_{yy} - \left[D_{12} - (1 - \nu)D\right]\dot{w}_{xx} = 0
\end{aligned}
\right\} \qquad (103)
$$

at $y = 0$ and $y = b$. The boundary conditions at $x = 0$ and $x = a$ are fulfilled automatically if the function $\dot{w}$ is of the form given by equations (80) and (81). Equations (103), applied to the function $f(y)$ at $y = 0$, require that

$$
\left.
\begin{aligned}
&c_1 + c_3 = 0 \\
&-D_{22}\left(r^2 c_1 - s^2 c_3\right) + \left(\frac{m\pi}{a}\right)^2 \left[D_{12} - (1 - \nu)D\right](c_1 + c_3) = 0
\end{aligned}
\right\} \qquad (104)
$$

Consequently,

$$
\left.
\begin{aligned}
&c_1 = -c_3 \\
&c_3\left(r^2 + s^2\right) = 0
\end{aligned}
\right\} \qquad (105)
$$

thus

$$c_1 = c_3 = 0 \qquad (106)$$

since $r^2 + s^2 \neq 0$ if buckling is to take place.

The conditions at $y = b$ imply that

$$
\left.\begin{aligned}
&c_2 \sinh rb + c_4 \sin sb = 0 \\
&-D_{22}\left(r^2 c_2 \sinh rb - s^2 c_4 \sin sb\right) \\
&\qquad + \left(\frac{m\pi}{a}\right)^2 \left[D_{12} - D(1 - \nu)\right]\left(c_2 \sinh rb + c_4 \sin sb\right) = 0
\end{aligned}\right\} \tag{107}
$$

which are equivalent to

$$
\left.\begin{aligned}
&c_2 \sinh rb + c_4 \sin sb = 0 \\
&r^2 c_2 \sinh rb - s^2 c_4 \sin sb = 0
\end{aligned}\right\} \tag{108}
$$

Now these equations are linear, homogeneous equations in the two unknowns $c_2$ and $c_4$; they possess nontrivial solutions if and only if

$$
\begin{vmatrix} \sinh rb & \sin sb \\ r^2 \sinh rb & -s^2 \sin sb \end{vmatrix} = -\left(r^2 + s^2\right) \sin sb \sinh rb = 0 \tag{109}
$$

This condition will be satisfied if $r^2 + s^2 = 0$, $r = 0$, or $s = n\pi$ where $n = 0, 1, 2, \ldots$. The first possibility is untenable; the second implies, according to equations (108), that $s = 0, \pi, 2\pi, \ldots$. If $s = 0$, then from equations (108) $r$ must also vanish and buckling will not take place. The buckling condition must therefore be $s = n\pi$ where $n = 1, 2, 3, \ldots$.

Since $s = \sqrt{p(q - p)}$ from equations (85), it follows from equations (82) that

$$s^2 = \frac{n^2\pi^2}{b^2}$$

$$= p(q - p)$$

$$= \frac{m\pi}{a}\sqrt{\frac{D_{12}}{D_{22}}}\left[\sqrt{\frac{\sigma_o h}{D_{12}} - \left(\frac{m\pi}{a}\right)^2 \frac{D_{11}D_{22} - D_{12}^2}{D_{11}D_{22}}} - \frac{m\pi}{a}\sqrt{\frac{D_{12}}{D_{22}}}\right] \tag{110}$$

With the notation of the previous sections for $D_{11}'$, $D_{12}'$, $D_{22}'$, and $\Delta$, this relation can be written as

$$\frac{n^2\pi^2}{b^2} = \frac{m\pi}{a}\sqrt{\frac{D_{12}'}{D_{22}'}}\left[\sqrt{\frac{\sigma_o h}{DD_{12}'} - \left(\frac{m\pi}{a}\right)^2 \frac{D_{22}'\Delta}{D_{12}'}} - \frac{m\pi}{a}\sqrt{\frac{D_{12}'}{D_{22}'}}\right] \tag{111}$$

The critical stress $\sigma_{cr}$ can then be found by direct computation to be

$$\sigma_{cr} = \sigma_o$$

$$= \frac{\pi^2 DD_{12}'}{b^2 h}\left(\frac{n^2}{m}\frac{a}{b}\sqrt{\frac{D_{22}'}{D_{12}'}} + m\frac{b}{a}\sqrt{\frac{D_{12}'}{D_{22}'}}\right)^2 + \frac{DD_{22}'\Delta}{h}\left(\frac{m\pi}{a}\right)^2 \tag{112}$$

Now each of the terms in the first parentheses on the right-hand side is positive; consequently, the minimum critical stress will occur when $n = 1$. Thus the critical stress is given by

$$\sigma_{cr} = \frac{\pi^2 D D_{12}'}{b^2 h}\left(\frac{1}{m}\frac{a}{b}\sqrt{\frac{D_{22}'}{D_{12}'}} + m\frac{b}{a}\sqrt{\frac{D_{12}'}{D_{22}'}}\right)^2 + \frac{D D_{22}'\Delta}{h}\left(\frac{m\pi}{a}\right)^2 \tag{113}$$

In the elastic case, $\lambda = 1$ and the functions appearing in the formula for the critical stress take on the values $D_{11}' = D_{12}' = D_{22}' = 1$, $\Delta = 0$; consequently,

$$\sigma_{cr} = \frac{\pi^2 D}{b^2 h}\left(\frac{1}{m}\frac{a}{b} + m\frac{b}{a}\right)^2 \tag{114}$$

This result is the same as that found for the elastic case directly (reference 1, p. 330). Timoshenko suggests (reference 1, p. 387) the application of the following equation for the buckling of a simply supported plate compressed beyond the elastic limit,

$$D\left(\alpha\dot{w}_{xxxx} + 2\sqrt{\alpha}\,\dot{w}_{xxyy} + \dot{w}_{yyyy}\right) + N_x\dot{w}_{xx} = 0 \tag{115}$$

where here $\alpha = E^*/E_0$, the ratio of the reduced modulus (defined in equation (100)) to Young's modulus. Under these assumptions, the critical compressive stress is found to be

$$\sigma_{cr} = \frac{\pi^2 D}{b^2 h}\left(\frac{1}{m}\frac{a}{b} + m\frac{b}{a}\sqrt{\alpha}\right)^2 \tag{116}$$

Although Timoshenko (reference 1, p. 387) gives only the critical stress for the half-wave number $m = 1$, his results can be easily extended to the form given in equation (116).

It will be found more convenient to use a new quantity $\bar{k}$ rather than $\sigma_{cr}$ for comparison of the two theories. This parameter is a modified form of $k$ defined in equation (94); more precisely,

$$\bar{k} = \frac{b^2}{a^2}k$$

$$= \frac{12\sigma_{cr}b^2}{\pi^2 h^2 E_o}\left(1 - \nu^2\right)$$

$$= \frac{\sigma_{cr}b^2 h}{\pi^2 D} \tag{117}$$

For the theory proposed in the present report, this quantity becomes

$$\bar{k} = D_{12}'\left(\frac{1}{mR}\sqrt{\frac{D_{22}'}{D_{12}'}} + mR\sqrt{\frac{D_{12}'}{D_{22}'}}\right)^2 + D_{22}'\Delta m^2 R^2 \tag{118}$$

where $R = b/a$, as before. In the elastic case, $\bar{k}$ takes the form

$$\bar{k} = \left(\frac{1}{mR} + mR\right)^2 \tag{119}$$

On the other hand, Timoshenko's result can be written as

$$\bar{k} = \left(\frac{1}{mR} + mR\sqrt{\alpha}\right)^2 \tag{120}$$

The differences in the two results appear more readily if equations (118) and (120) are expanded in full. After some simplifications, equation (118) becomes

$$\bar{k} = D_{11}'m^2R^2 + 2D_{12}' + \frac{D_{22}'}{m^2R^2} \tag{121}$$

while equation (120) takes the form

$$\bar{k} = \alpha m^2R^2 + 2\sqrt{\alpha} + \frac{1}{m^2R^2} \qquad (122)$$

It is seen then that $\bar{k}$ is the same type of rational function of $m^2R^2$ for both theories except for the fact that the coefficients are different functions of $\lambda$.

Unlike in the case of the cruciform section which is treated in the next section, the computations involved in evaluating $\bar{k}$ as a function of $\lambda$ and $mR$ are relatively simple since the coefficients $D_{11}'$, $D_{12}'$, $D_{22}'$, and $\alpha$ can be tabulated once for all independently of the stress-strain law and the particular geometrical ratios under consideration. The critical values of $\bar{k}$ can be found with reasonable speed by the following procedure. Curves of $\bar{k}$ against $\lambda$ can be obtained for various values of the parameters $m$ and $R$ by evaluating equation (121) or (122), depending on which theory is used. These results, which form the bulk of the computation, do not depend on the stress-strain law but only on $\nu$. Consequently, the curves of $\bar{k}$ against $\lambda$ thus found are valid for all materials having the same Poisson's ratio. On the other hand, $\bar{k}$ and $\lambda$ are related by a second equation which depends on the stress-strain law; namely, from equation (117),

$$\bar{k} = \frac{12\sigma_{cr}b^2}{\pi^2h^2E_o}\left(1 - \nu^2\right)$$

Since $\sigma_{cr}$ is a function of $\lambda$, determined by the given compressive stress-strain law of the material in question, $\bar{k}$ also can be plotted as a function of $\lambda$ once the plate width-to-thickness ratio $b/h$ has been fixed for a particular example. If these curves are plotted on the same sheet as those previously described, the intersections will give the critical value of $\bar{k}$, and hence the critical stress, corresponding to a given stress-strain law and given plate parameters $b/h$ and $R$. It should be noted that $\bar{k}$ depends on the ratio $b/h$ in the plastic range, whereas this is not the case within the ordinary elastic theory.

Since the quantity $\lambda$ enters into the computations for the second set of curves through the stress-strain relations, it is quite useful to represent the stress-strain curve analytically. An expression of the form

$$\epsilon = \frac{1}{E_o}\sigma + A\sigma^{\beta} \tag{123}$$

has been fitted to the experimental data in such a way that the experimental and theoretical curves pass through the same initial and final points and the slopes at these points coincide. The fitted curve together with the experimental points is shown in figure 5.

The computational program described has been carried out for the material given in figure 5 and the ratio of plate width to plate thickness fixed at $b^2/h^2 = 1000$. The solid curves of figure 6 give $\bar{k}$ as a function of $\lambda$ with parameter R as defined by equation (121), each set of curves representing a different value of m. These curves can be applied to a plate with any ratio of b/h and any compressive stress-strain law provided $\nu = 0.32$. The dashed lines, on the other hand, represent the relation between $\bar{k}$ and $\lambda$ given by equation (117); they correspond to $b^2/h^2 = 1000$ and the stress-strain curve of figure 5. The points of intersection have been determined and the critical values of $\bar{k}$ plotted in the usual fashion as functions of $1/R$ in figure 7. Timoshenko's results are also shown as well as those for the elastic theory. Since the critical stress is given by $\bar{k}$ multiplied by a constant, the interpretation of these curves of $\bar{k}$ against $1/R$ holds equally well for the critical stress as a function of $1/R$.

First, it should be noted that the plate will buckle in one or more half waves according to the magnitude of the ratio $1/R = a/b$; this holds for plastic buckling as well as elastic buckling. Furthermore, it is necessary only to consider that part of the curve corresponding to a given value of m which lies below the intersections with curves belonging to adjacent values of m. Under these conditions, it can be readily seen that the critical stress obtained from the present theory lies between the results of the elastic analysis and Timoshenko's formal procedure. More precisely, the elastic critical stress is higher than that predicted by the present theory, whereas Timoshenko's buckling stress is lower. The transition from a buckling mode with a given number of half waves to the next higher occurs at practically the same values of $1/R$ in both plastic theories; whereas, this transition occurs at slightly larger values of $1/R$ in the elastic range. Finally, the locus of minimums of all three sets of curves is, to a high degree of approximation, a straight line. While this straight line is fixed for all values of $b/h$ in the elastic case, it will shift in the plastic case as this parameter is changed.

With a slightly different computational procedure, the results just obtained for the buckling of a simply supported plate by the present theory can be compared with Ilyushin's solution of the same problem (reference 6). Ilyushin's general equation for the buckling of a rectangular plate compressed in one direction is of the same form as equation (72) except for the difference in the coefficients. His fundamental relation can be written as

$$D_{11}{}^{*}\dot{w}_{xxxx} + 2D_{12}{}^{*}\dot{w}_{xxyy} + D_{22}{}^{*}\dot{w}_{yyyy} = -h\sigma_{o}\dot{w}_{xx} \qquad (124)$$

where the coefficients $D_{11}{}^{*}$, $D_{12}{}^{*}$, and $D_{22}{}^{*}$ are rather complicated functions of the stresses which will be discussed shortly. Since these results apply only to incompressible materials, the theory developed in the present report must be specialized to the case for which $\nu = \frac{1}{2}$.

For the case of a simply compressed plate, the basic functions entering in the definition of the coefficients in equation (124) can be given in a slightly less complicated manner than that used by Ilyushin for the more general problem. Let $E_s$ denote the "secant modulus," that is, quotient of the compressive stress divided by the compressive strain as obtained from a compression test for the material in question. Then Ilyushin's function $\omega$ can be shown to be

$$\omega = 1 - \frac{E_s}{E_o} \qquad (125)$$

where $E_o$ is Young's modulus for the material. With Von Kármán's modulus, $E^*$, as given in equation (100), a new quantity $\kappa$ may be defined by setting

$$\kappa = E^*/E_o \qquad (126)$$

The function $\psi$ will be used to designate the following relation containing $\kappa$ and $\omega$,

$$\psi = \omega\left(1 - \frac{1}{2}\sqrt{\kappa}\right)\left[\left(1 - \frac{1}{2}\sqrt{\kappa}\right)^2 + \frac{3}{4}\frac{\kappa}{1 - \left(1 - \frac{1}{2}\sqrt{\kappa}\right)\omega}\right] \tag{127}$$

The coefficients $D_{11}^*$, $D_{12}^*$, and $D_{22}^*$ can then be shown to be of the form

$$\left.\begin{aligned} D_{11}^* &= D(1 - \psi)\left(1 - \frac{3}{4}\frac{1 - \psi - \kappa}{1 - \psi}\right) \\ D_{12}^* &= D(1 - \psi) \\ D_{22}^* &= D(1 - \psi) \end{aligned}\right\} \tag{128}$$

where D is the flexural rigidity of the plate defined in equations (73). It should be noted here that Ilyushin's coefficients depend on the two moduli $E$ and $E_s$, whereas those appearing in the present theory depend only on the tangent modulus $E$ or rather the ratio $\lambda = E/E_0$. Consequently, it is extremely difficult to carry out the larger part of the computations for $\bar{k}$ independently of the stress-strain law as can be done with the method presented herein. The numerical technique must therefore be changed somewhat.

The boundary conditions for the simply supported plate in Ilyushin's analysis are

$$\left.\begin{aligned} \dot{w} &= 0 \\ -D_{11}^*\dot{w}_{xx} - \frac{1}{2}D_{22}^*\dot{w}_{yy} &= 0 \end{aligned}\right\} \tag{129}$$

at $x = 0$ and $x = a$, and

$$\left.\begin{aligned} \dot{w} &= 0 \\ -\frac{1}{2}D_{22}{}^*\dot{w}_{xx} - D_{22}{}^*\dot{w}_{yy} &= 0 \end{aligned}\right\} \tag{130}$$

at $y = 0$ and $y = b$. These boundary conditions are completely analogous to equations (102) and (103), provided Poisson's ratio $\nu$ is taken to be 1/2 in the equations (102) and (103). If the new quantities $\bar{D}_{11}$, $\bar{D}_{12}$, and $\bar{D}_{22}$ are introduced in the following manner

$$\left.\begin{aligned} \bar{D}_{11} &= D_{11}{}^*/D \\ \bar{D}_{12} &= D_{12}{}^*/D \\ \bar{D}_{22} &= D_{22}{}^*/D \end{aligned}\right\} \tag{131}$$

the critical parameter $\bar{k}$ can be found in precisely the same fashion as that used in developing equation (121). Consequently,

$$\bar{k} = \bar{D}_{11}m^2R^2 + 2\bar{D}_{12} + \frac{\bar{D}_{22}}{m^2R^2} \tag{132}$$

As has been previously noted, the coefficients $\bar{D}_{11}$, $\bar{D}_{12}$, and $\bar{D}_{22}$ are functions of both the secant modulus and the tangent modulus with the result that specific reference to the stress-strain law in compression must be made in order to evaluate equation (132) readily. The critical quantity $\bar{k}$ has been computed as a function of $\sigma_{cr}$ for the material shown in figure 5 with $m$ and $R$ appearing as parameters. For a given width-to-thickness ratio $b/h$, $\bar{k}$ can be determined by finding the intersections of these curves with the straight line defined by equation (117). This procedure has been applied to the case $b^2/h^2 = 1000$ previously discussed. The results obtained from Ilyushin's method and from the present method with $\nu = \frac{1}{2}$ are shown in figure 8. Again considering only those parts of the curves which lie below the intersection points corresponding to consecutive values of $m$, certain general conclusions can be drawn. The minimum value of $\bar{k}$ as determined by Ilyushin's method is practically constant for all values of $m$ as has been seen in the other theories considered. On the other hand, Ilyushin's theory predicts smaller values of $\bar{k}$ and hence lower critical stresses than those of the present theory. Comparison with figure 7 would indicate that Ilyushin's critical stresses lie roughly between those of Timoshenko and the present theory. Finally, buckling occurs at practically the same wave number in both theories although the jump in form takes place at slightly smaller values of $1/R$ in the present theory.

Torsional buckling of cruciform sections.- The torsional buckling of a cruciform section under compression can be studied by treating each flange as a simply compressed plate, simply supported at $y = 0$ and free at $y = b$. (See reference 1, p. 340.) As in the previous case, the deflection rate $\dot{w}$ will be given by equation (80) where $f(y)$ is found from equation (84). The conditions for the simply supported edge (equations (75)) at $y = 0$ become

$$\left.\begin{aligned} c_3 &= -c_1 \\ f''(0) &= 0 \end{aligned}\right\} \tag{133}$$

The last equation implies $r^2c_1 = s^2c_3$. Since $c_3 = -c_1$ and $r^2$ and $s^2$ do not vanish, $c_1 = c_3 = 0$; consequently,

$$f(y) = c_2 \sinh ry + c_4 \sin sy \tag{134}$$

Finally, the boundary conditions at the free edge $y = b$ require, according to equations (77), that

$$\left.\begin{aligned} -\left(\frac{m\pi}{a}\right)^2 \Big[D_{12} - (1 - \nu)D\Big]\left(c_2 \sinh rb + c_4 \sin sb\right) & \\ + D_{22}\left(r^2c_2 \sinh rb - s^2c_4 \sin sb\right) &= 0 \\ -\left(\frac{m\pi}{a}\right)^2 \Big[D_{12} + (1 - \nu)D\Big]\left(rc_2 \cosh rb + sc_4 \cos sb\right) & \\ + D_{22}\left(r^3c_2 \cosh rb - s^3c_4 \cos sb\right) &= 0 \end{aligned}\right\} \tag{135}$$

Now equations (135) are linear homogeneous equations in the two unknowns $c_2$ and $c_4$. If these equations are to yield nonvanishing solutions, the determinant of the coefficients must vanish; that is,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \tag{136}$$

where

$$a_1 = \left(D_{22}r^2 - \left\{m^2\pi^2\left[D_{12} - (1 - \nu)D\right]/a^2\right\}\right) \sinh rb$$

$$b_1 = -\left(D_{22}s^2 + \left\{m^2\pi^2\left[D_{12} - (1 - \nu)D\right]/a^2\right\}\right) \sin sb$$

$$a_2 = r\left(D_{22}r^2 - \left\{m^2\pi^2\left[D_{12} + (1 - \nu)D\right]/a^2\right\}\right) \cosh rb$$

$$b_2 = -s\left(D_{22}s^2 + \left\{m^2\pi^2\left[D_{12} + (1 - \nu)D\right]/a^2\right\}\right) \cos sb$$

Except for a systematic interchange of the hyperbolic and trigonometric functions, the determinant in equation (136) is the same as that appearing in equation (90). The determinantal equation can then be reduced to

$$s\left\{D_{22}r^2 - \frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\right\}^2 \tanh rb$$
$$= r\left\{D_{22}s^2 + \frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\right\}^2 \tan sb \tag{137}$$

With the notation introduced in equations (92), this result can be rewritten in the form

$$\left(\xi^2 - m^2\pi^2R^2Q\right)^2 \frac{\tanh \xi}{\xi} = \left(\eta^2 + m^2\pi^2R^2Q\right)^2 \frac{\tan \eta}{\eta} \qquad (138)$$

Again $\xi$ and $\eta$ can be determined as functions of the quantities $D_{11}'$, $D_{12}'$, $D_{22}'$, and $k$ defined in equations (94). With some additional transformations, equation (138) can then be put into a form suitable for solving for the quantity $\overline{k}$ as a function of $\lambda$. Although the computations are somewhat tedious, the solutions obtained are completely independent of the stress-strain law. These results are represented by the solid lines of figure 9 and are marked for various values of the side ratio $R$. The dashed curves, on the other hand, represent the quantity $\overline{k}$ as a function of $\lambda$ as derived from equation (117). These curves have been computed for the stress-strain law shown in figure 5 and the values of the width-to-thickness ratio $b/h$ as indicated in the graph. Given the two dimensionless quantities $b/a$ and $b/h$, the critical stress is then determined by the corresponding solid curve and dashed curve. This point gives the desired value of $\overline{k}$, from which the buckling stress is found by solving equation (117). The details of the procedure are found in appendix C. It should also be pointed out that the solid curves have been computed for the half-wave number $m = 1$, for it is shown in appendix C that the lowest value of $\overline{k}$, and consequently the lowest critical stress, will be obtained for this value of the wave number.

Certain general conclusions concerning the buckling of such a section can be reached without reference to the stress-strain law by means of the solid curves of figure 9. These curves show that, for a given side ratio $R$, $\overline{k}$ is a decreasing function of $\lambda$. This decrease is not the same for all values of the side ratio, however. For small values of $R$ (large length-to-width ratios), $\overline{k}$ is almost constant. In fact, when $R \leqq 0.10$, $\overline{k}$ is practically constant and the critical stress is the same as that found in the theory of elastic buckling of plates. The plastic effects are very pronounced for larger values of $R$ and increase as $R \longrightarrow 1$, that is, as the rectangular plate becomes more nearly square.

Finally, the intersections of the solid and dashed curves of figure 9 yield $\overline{k}$ as a function of the side ratio $R$; these are replotted in figure 10. The top curve represents the elastic case ($\lambda = 1$) and is independent of $b/h$. The other three curves correspond to the cases $b^2/h^2 = 300$, 250, and 200, respectively. It should be noted that $\overline{k}$ for the elastic case is always greater than $\overline{k}$ for the plastic case; this difference is more marked for shorter (larger $R$) and thicker (smaller $b^2/h^2$) plates.

## Energy Method

In the preceding section rigorous solutions were obtained for several cases of plastic buckling of simply compressed plates. In cases for which the rigorous solution becomes too unwieldy or is not known at all, the energy method provides a convenient means of finding approximate values of the buckling loads.

As far as the elastic buckling of plates is concerned, the energy method is well established. (See, for instance, reference 1, pp. 325 ff.) At first glance, it may appear somewhat doubtful whether this method can be extended to the plastic buckling of plates. That such an extension is legitimate, however, follows from the fact that equation (109) has the same form as the equation for the elastic buckling of an orthotropic plate with flexural rigidities $D_{11}$ and $D_{22}$ and torsional rigidity $D_{12}$. Since the energy method is valid for this problem of elastic buckling and since the plastic buckling of a simply compressed plate is governed by the same equation, the application of the energy method to the plastic buckling of plates is legitimate.

Under sufficiently small edge thrusts the flat form of the plate represents a stable equilibrium configuration. When the edge thrusts reach the critical value, however, this equilibrium becomes indifferent and the plate may assume a bent form. This transition from the flat to the bent form, that is, from one indifferent equilibrium configuration to another one, does not involve any energy input or output. Accordingly, the work done by the edge thrusts equals the flexural energy of the bent plate. The mathematical expression of this principle is the basic energy equation:

$$D \iint \left\{ \left[1 - c\delta(2 - \nu)^2\right] \dot{w}_{xx}^2 + 2\left[\nu - c\delta(2 - \nu)(2\nu - 1)\right] \dot{w}_{xx}\dot{w}_{yy} + \left[1 - c\delta(2\nu - 1)^2\right] \dot{w}_{yy}^2 + 2(1 - \nu)\dot{w}_{xy}^2 \right\} dx\, dy = \sigma_0 h \iint \dot{w}_x^2\, dx\, dy \tag{139}$$

where the range of integration is the area of the plate, that is, $0 \leqq x \leqq a$, $0 \leqq y \leqq b$. As is easily seen from equations (46) and (74), the left-hand side of equation (139) equals

$$-E_o \iint \left( \dot{M}_x \dot{K}_{11} - \dot{M}_{xy} \dot{K}_{12} + \dot{M}_y \dot{K}_{22} \right) dx\, dy \tag{140}$$

which represents twice the rate at which work is done by the bending and twisting moments. In the elastic range, $c = 0$, this expression can be written in the familiar form

$$D \iint \left[ \left( \dot{w}_{xx} + \dot{w}_{yy} \right)^2 - 2(1 - \nu) \left( \dot{w}_{xx} \dot{w}_{yy} - \dot{w}_{xy}{}^2 \right) \right] dx\, dy \tag{141}$$

The right-hand side of equation (139) represents twice the rate at which work is done by the edge thrusts and has the same form as in the elastic case.

The following expression for the critical load of the plate is obtained from equation (139):

$$\sigma_o h = \frac{\overline{A}}{B} \tag{142}$$

where

$$\begin{aligned} \overline{A} = D \iint \Big\{ & \left[ 1 - c\delta(2 - \nu)^2 \right] \dot{w}_{xx}{}^2 + 2 \left[ \nu - c\delta(2 - \nu)(2\nu - 1) \right] \dot{w}_{xx} \dot{w}_{yy} \\ & + \left[ 1 - c\delta(2\nu - 1) \right] \dot{w}_{yy}{}^2 + 2(1 - \nu) \dot{w}_{xy}{}^2 \Big\} dx\, dy \end{aligned} \tag{143}$$

and

$$B = \iint \dot{w}_x{}^2\, dx\, dy \tag{144}$$

The right-hand side of equation (142) depends on the function $\dot{w}$, that is, on the deflected shape of the plate. Now, any restriction which is imposed on the deflection rate $\dot{w}$, over and above the boundary conditions discussed in the third section of the ANALYSIS amounts to an increase in the stiffness of the plate and must, therefore, lead to an increase in the critical load. The critical load for a given plate is accordingly found as the smallest value which the right-hand side of equation (142) can assume for functions $\dot{w}$ possessing continuous partial derivatives of the second order and satisfying the boundary conditions for the deflection of the plate.

In the elastic case, $c = 0$, the right-hand side of equation (142) is independent of the buckling stress $\sigma_o$. Equation (142) therefore furnishes the buckling stress $\sigma_o$ which corresponds to a given plate thickness $h$. In the plastic case, however, the numerator $\overline{A}$ of the right-hand side of equation (142) depends on the buckling stress $\sigma_o$ through the quantities $c$ and $\delta$. Equation (142) should therefore be considered as an equation which determines the critical thickness $h$ corresponding to a given buckling stress. If the expression for $D$, from equations (73), is substituted into equation (143) and the resulting form of equation (142) solved with respect to $h$, the following formula is obtained:

$$h = \sqrt{\frac{12\sigma_o(1 - \nu^2)B}{E_oA'}} \tag{145}$$

where $B$ is given by equation (144) and $A'$ denotes the integral on the right-hand side of equation (143).

The energy method will now be applied to the buckling of a cruciform section previously considered. A suitable choice of the approximate deflection rate $\dot{w}$ is

$$\dot{w} = \eta\left(\xi - 2\xi^3 + \xi^4\right) \tag{146}$$

where $\xi = x/a$ and $\eta = y/b$. This function satisfies all the boundary conditions on the plate except at the free edge $y = b$. Nevertheless, this expression proves to be a satisfactory approximation as will be seen shortly. The necessary integrations indicated in the definitions of $\overline{A}$ and $B$ (equations (143) and (144)) can be easily carried out in this case; for example,

$$\begin{aligned}\iint \dot{w}_{xx}^2 \, dx \, dy &= \frac{144R}{a^2}\int_0^1\int_0^1 \eta^2\left(\xi - \xi^2\right) d\xi \, d\eta \\ &= \frac{1.6R}{a^2}\end{aligned} \tag{147}$$

$$\iint \dot{w}_{xy}^{2}\, dx\, dy = \frac{1}{a^{2}R}\int_{0}^{1}\left(1 - 12\xi^{2} + 8\xi^{3} + 36\xi^{4} - 48\xi^{5} + 16\xi^{6}\right) d\xi\, d\eta$$

$$= \frac{17}{35a^{2}R} \tag{148}$$

$$\iint \dot{w}_{x}^{2}\, dx\, dy = R\int_{0}^{1}\int_{0}^{1} \eta^{2}\left(1 - 12\xi^{2} + 8\xi^{3} + 36\xi^{4} - 48\xi^{5} + 16\xi^{6}\right) d\xi\, d\eta$$

$$= \frac{17R}{105} \tag{149}$$

where $R = b/a$. Thus

$$\frac{B}{A'} = \frac{17a^{2}R^{2}}{168R^{2}\left[1 - c\delta(2 - \nu)^{2}\right] + 102(1 - \nu)}$$

$$= \frac{17a^{2}R^{2}}{168R^{2}D_{11}' + 102(1 - \nu)} \tag{150}$$

and

$$\overline{k} = \frac{168R^{2}D_{11}' + 102(1 - \nu)}{17\pi^{2}} \tag{151}$$

where $\overline{k}$ is defined in equation (117).

Since $D_{11}'$ has been previously computed as a function of $\lambda$ (see table I), $\overline{k}$ may be readily evaluated from equation (151) as a function of $\lambda$ for various values of the parameter $R$. This has been done for the example $\nu = 0.32$ considered in an earlier section.

The results are shown in figure 11 in which the solid lines represent the solution obtained by the energy method and the dashed lines are those found from the exact solution. A brief inspection will show that the energy method, as applied here, gives a very good approximation in the technically interesting region of small values of R. The error increases as R becomes large and reaches a maximum of about 8 percent. It should also be noted that for a given value of R the error is an increasing function of $\lambda$. The energy method actually yields better results when the complete problem of determining the critical values of $\bar{k}$ associated with a given stress-strain law is carried out. These points are found from the intersections of the curves of figure 11 and the curves derived from the stress-strain law through equation (117) (fig. 9). Since the curves of figure 9 are monotonic, increasing functions of $\lambda$ with a slope angle of less than 90°, the error will be smaller than the original estimate. The results of applying this method to the cases treated in the previous section on exact analysis are shown in figure 12. The solid curves represent the solution by the energy method, whereas the dashed curves are the exact solution. It is seen that the maximum error is of the order of $2\frac{1}{2}$ percent of the value of $\bar{k}$. This appears to be well within the order of accuracy of the theory itself. Finally, the approximate value of $\bar{k}$ is always greater than the exact solution as is usually found in the application of the energy method.

Equation (151) also affords an easy means of studying the general character of the critical parameter $\bar{k}$ as a function of $\lambda$ for various values of R. It has already been pointed out that $D_{11}'$ is a decreasing function of $\lambda$ starting at $D_{11}' = 1$ for $\lambda = 1$ and approaching $D_{11}' = 0.2413$ for $\lambda \longrightarrow \infty$. Thus for small values of R, the first term has little influence and $\bar{k}$ is approximately a horizontal line defined by the equation

$$\bar{k} = \frac{102(1 - \nu)}{17\pi^2} \tag{152}$$

The elastic solution based on the energy method with the deflection given by equation (146) is

$$\bar{k} = \frac{168R^2 + 102(1 - \nu)}{17\pi^2} \tag{153}$$

and equations (152) and (153) agree quite closely for small values of R. Under these circumstances, then, the critical stress determined from the plastic theory will be the same as that obtained from the elastic formulas.

Brown University
Providence, R. I., March 26, 1947

# APPENDIX A

## DIFFICULTIES PRESENTED BY ILYUSHIN'S STRESS-STRAIN RELATIONS

It has been pointed out in the first section of the ANALYSIS that certain objectionable discontinuities are present at the neutral surface when the state of stress and strain in a body is described by relations of the type used by Ilyushin (reference 6). More precisely, it has been shown that the equations for loading and unloading do not give the same increments in strain at the neutral surface; this difficulty is not present in the theory of plastic flow developed in the present report. Such problems always arise when an attempt is made to take unloading into account in a theory of plastic deformation. (See reference 10, p. 400.)

The inconsistency is brought out more clearly if the whole problem is considered in terms of the state of stress and strain existing in the buckled plate. Let

$$\Omega = \Omega(\sigma_i) = \frac{\omega}{1 - \omega} \tag{A1}$$

Then Ilyushin's stress-strain relations for loading in the case under consideration (equations (30)) can be written in the form

$$\left.\begin{aligned} 2E_o\epsilon_x &= (1 + \Omega)(2\sigma_x - \sigma_y) \\ 2E_o\epsilon_y &= (1 + \Omega)(-\sigma_x + 2\sigma_y) \\ E_o\gamma_{xy} &= 3(1 + \Omega)\tau_{xy} \end{aligned}\right\} \tag{A2}$$

Differentiated with respect to time, these become equations of the following type:

$$2E_o\dot{\epsilon}_x = (1 + \Omega)(2\dot{\sigma}_x - \dot{\sigma}_y) + \dot{\Omega}(2\sigma_x - \sigma_y) \tag{A3}$$

In particular, for buckling from a state of simple compression, $\sigma_x = -\sigma_o$ and $\sigma_y = \tau_{xy} = 0$,

$$\begin{aligned}\dot{\Omega} &= \frac{d\Omega}{d\sigma_i}\dot{\sigma}_i \\ &= -\frac{1}{2}\Omega'\left(2\dot{\sigma}_x - \dot{\sigma}_y\right)\end{aligned} \tag{A4}$$

where $\Omega' = d\Omega/d\sigma_i$. Accordingly,

$$2E_o\dot{\epsilon}_x = \left(1 + \Omega + \sigma_o\Omega'\right)\left(2\dot{\sigma}_x - \dot{\sigma}_y\right) \tag{A5}$$

The boundary between the regions of loading and unloading is a plane $z$ = Constant. If the displacements are to be continuous across this plane, $\dot{\epsilon}_x$, $\dot{\epsilon}_y$, and $\dot{\gamma}_{xy}$, too, must be continuous. Moreover, according to Ilyushin's theory the sign of $-\sigma_o\dot{\epsilon}_x$ serves as a criterion for loading and unloading. Since $\dot{\epsilon}_x$ is continuous, it must vanish at the boundary between the regions of loading and unloading. Thus, $2\dot{\sigma}_x - \dot{\sigma}_y$ is continuous, too, and $\dot{\Omega} = 0$ on the boundary. Denoting the values on the loading side of the boundary by the superscript + and the values on the other side by the superscript -, the strains at the neutral surface become

$$\left.\begin{aligned}2E_o\dot{\epsilon}_x &= (1 + \Omega)\left(2\dot{\sigma}_x^{+} - \dot{\sigma}_y^{+}\right) \\ &= 2\dot{\sigma}_x^{-} - \dot{\sigma}_y^{-} \\ &= 0 \\ 2E_o\dot{\epsilon}_y &= (1 + \Omega)\left(-\dot{\sigma}_x^{+} + 2\dot{\sigma}_y^{+}\right) \\ &= -\dot{\sigma}_x^{-} + 2\dot{\sigma}_y^{-} \\ E_o\dot{\gamma}_{xy} &= 3(1 + \Omega)\dot{\tau}_{xy}^{+} \\ &= 3\dot{\tau}_{xy}^{-}\end{aligned}\right\} \tag{A6}$$

Thus

$$\left.\begin{aligned} \dot{\sigma}_x^{\,-} &= (1 + \Omega)\dot{\sigma}_x^{\,+} \\ \dot{\sigma}_y^{\,-} &= (1 + \Omega)\dot{\sigma}_y^{\,+} \\ \dot{\tau}_{xy}^{\,-} &= (1 + \Omega)\dot{\tau}_{xy}^{\,+} \end{aligned}\right\} \qquad (A7)$$

Since $(1 + \Omega) > 1$, the rates of stress on the unloading side of the boundary are of the same sign as those on the loading. Furthermore, the absolute values of the rates of stress on the unloading side are greater than those on the loading side. While this prediction of Ilyushin's stress-strain relations does not violate the equations of equilibrium, it does seem quite strange when compared with the usual notions of loading and unloading. The result is of a sufficiently startling character to call for direct experimental verification before a theory of structural stability in the plastic range is based on these stress-strain relations.

That the discontinuities implied in Ilyushin's stress-strain relations are a source of concern to Russian scientists working in this field is seen from the following passage which is quoted from W. W. Sokolovsky's recent paper (reference 14) in which the stability problem is specifically discussed:

"In solving these problems, the Mises-Hencky theory was employed, under the assumption of the incompressibility of the plastic material. This assumption has a strong influence upon the value of displacements and <u>does not permit satisfying completely the conditions of the continuity of all components of stress and displacements on the boundary between the elastic and the plastic zones</u>. An objective of recent work is to avoid these defects of the theory."

While the discontinuities appear to have been noticed, the source has not been traced correctly, for these discontinuities subsist even if the compressibility of the material is taken into account. They can be avoided only by replacing Ilyushin's theory of plastic deformation by a theory of plastic flow of the type given in the present report.

# APPENDIX B

## DETAILS IN DEVELOPMENT OF FUNDAMENTAL PLATE EQUATION

### Determination of Neutral Surface

The neutral surface can be determined from the condition that the additional stress resultants present in the buckled state must be in equilibrium. Rather than compute the rates of the stress resultants arising from $\dot{\sigma}_x$ and $\dot{\sigma}_y$ separately, it is much simpler to consider the combinations $\dot{\sigma}_x - \nu\dot{\sigma}_y$ and $\dot{\sigma}_y - \nu\dot{\sigma}_x$. (A similar device is employed in reference 6, p. 341.) For unloading, equations (40) may be used to yield

$$\left.\begin{aligned} \dot{\sigma}_x - \nu\dot{\sigma}_y &= \dot{\epsilon}_x \\ \dot{\sigma}_y - \nu\dot{\sigma}_x &= \dot{\epsilon}_y \end{aligned}\right\} \qquad \text{(B1)}$$

On the other hand, equations (39) give for loading

$$\left.\begin{aligned} \dot{\epsilon}_x &= \dot{\sigma}_x - \nu\dot{\sigma}_y + (\lambda - 1)\dot{\sigma}_x - \frac{\lambda - 1}{2}\dot{\sigma}_y \\ \dot{\epsilon}_y &= \dot{\sigma}_y - \nu\dot{\sigma}_x - \frac{\lambda - 1}{2}\dot{\sigma}_x + \frac{\lambda - 1}{4}\dot{\sigma}_y \end{aligned}\right\} \qquad \text{(B2)}$$

Transposition in the first of equations (B2) leads to

$$\dot{\sigma}_x - \nu\dot{\sigma}_y = \dot{\epsilon}_x - \frac{\lambda - 1}{2}\left(2\dot{\sigma}_x - \dot{\sigma}_y\right) \qquad \text{(B3)}$$

which, according to equations (41), can be written as

$$\dot{\sigma}_x - \nu\dot{\sigma}_y = \dot{\epsilon}_x - \frac{2(\lambda - 1)}{(5 - 4\nu)\lambda - (1 - 2\nu)^2}\left[(2 - \nu)\dot{\epsilon}_x + (2\nu - 1)\dot{\epsilon}_y\right] \qquad \text{(B4)}$$

With

$$c = \frac{\lambda - 1}{(5 - 4\nu)\lambda - (1 - 2\nu)^2} \tag{B5}$$

and $\dot{K}$ and $z_o$ as defined in equations (49), equation (B4) becomes

$$\dot{\sigma}_x - \nu\dot{\sigma}_y = \dot{\epsilon}_x + 2c\dot{K}(z - z_o) \tag{B6}$$

The quantity $c$ depends only on the properties of the material through $\nu$ and on the compressive stress through $\lambda$. It is a nondecreasing function of $\lambda$, for

$$\begin{aligned}
\frac{dc}{d\lambda} &= \frac{(5 - 4\nu)\lambda - (1 - 2\nu)^2 - (5 - 4\nu)(\lambda - 1)}{\left[(5 - 4\nu)\lambda - (1 - 2\nu)^2\right]^2} \\
&= \frac{5 - 4\nu - 1 + 4\nu - 4\nu^2}{\left[(5 - 4\nu)\lambda - (1 - 2\nu)^2\right]^2} \\
&= \frac{4(1 - \nu^2)}{\left[(5 - 4\nu)\lambda - (1 - 2\nu)^2\right]^2} \\
&\geqq 0
\end{aligned} \tag{B7}$$

since $-1 \leqq \nu \leqq \frac{1}{2}$. For $\lambda = 1$ (the smallest value of $\lambda$), $c = 0$; for $\lambda = \infty$, $c = 1/(5 - 4\nu)$. A similar expression for $\dot{\sigma}_y - \nu\dot{\sigma}_x$ can be found by the same technique; namely,

$$\dot{\sigma}_y - \nu\dot{\sigma}_x = \dot{\epsilon}_y - c\dot{K}(z - z_o) \tag{B8}$$

The rates of change of the stress resultants $\dot{N}_x$ and $\dot{N}_y$ can be computed from equations (B1), (B6), and (B8). Two cases arise depending on whether the region $z > z_o$ is a region of loading or whether $z < z_o$ is the region of loading, that is, whether $\dot{K} > 0$ or whether $\dot{K} < 0$.

Case (1), $\dot{K} > 0$.- Since $\dot{K} > 0$, the region $z > z_o$ is the region of loading and $z < z_o$ that of unloading. Consequently, equations (B1) are valid for $z \leq z_o$ and equations (B6) and (B8) hold for $z \geq z_o$. Now the rates of the stress resultants $\dot{N}_x$ and $\dot{N}_y$ are defined as

$$\left.\begin{aligned} \dot{N}_x &= \int_{-h/2}^{h/2} \dot{\sigma}_x \, dz \\ \dot{N}_y &= \int_{-h/2}^{h/2} \dot{\sigma}_y \, dz \end{aligned}\right\} \tag{B9}$$

According to equations (B1) and (B6) then

$$\dot{N}_x - \nu\dot{N}_y = \int_{-h/2}^{h/2} \dot{\epsilon}_x \, dz + 2c\dot{K} \int_{z_o}^{h/2} \left(z - z_o\right) dz \tag{B10}$$

The strain rate $\dot{\epsilon}_x = \dot{\epsilon}_1 - z\dot{K}$ from equations (45); neither the function $\dot{\epsilon}_1$ nor $\dot{K}$ depends on $z$. Therefore

$$\begin{aligned} \int_{-h/2}^{h/2} \dot{\epsilon}_x \, dz &= \int_{-h/2}^{h/2} \dot{\epsilon}_1 \, dz - \dot{K} \int_{-h/2}^{h/2} z \, dz \\ &= \dot{\epsilon}_1 h \end{aligned} \tag{B11}$$

and

$$\dot{N}_x - \nu\dot{N}_y = \dot{\epsilon}_1 h + c\dot{K}\left(\frac{h}{2} - z_o\right)^2 \tag{B12}$$

In the same way, it can be shown that

$$\dot{N}_y - \nu\dot{N}_x = \int_{-h/2}^{h/2} \dot{\epsilon}_y \, dz - c\dot{K}\int_{z_o}^{h/2} \left(z - z_o\right) dz$$

$$= \dot{\epsilon}_2 h - \frac{1}{2} c\dot{K}\left(\frac{h}{2} - z_o\right)^2 \tag{B13}$$

From equations (B11) and (B13) it has been shown in the second section of the ANALYSIS that the position of the middle surface is given by that value of $z_o$ which is the solution of equation (58). With $\zeta_o = 2z_o/h$, equation (58) becomes equation (60); and with equation (62) the solution can be written as

$$\zeta_o = \alpha \pm \sqrt{\alpha^2 - 1} \tag{B14}$$

Since $c$ is a nondecreasing function of $\lambda$ with values in the range $0 \leqq c \leqq 1/(5 - 4\nu)$, the quantity $\alpha$ will lie in the range

$$-\infty \leqq \alpha \leqq -1 \tag{B15}$$

Consequently, only the positive root in equation (B14) can be kept if the inequality $\left|\zeta_o\right| = \left|2z_o/h\right| \leqq 1$ is to be preserved. It will be convenient to denote this root by $\zeta_o^{\ +}$; that is,

$$\zeta_o^{\ +} = \zeta_o = \alpha + \sqrt{\alpha^2 - 1} \tag{B16}$$

This is the formula for the neutral surface given in the second section of the ANALYSIS.

Case (2), $\dot{K} < 0$.- The same technique, as used for case (1), with the exception of certain changes in sign can be employed in the case when $\dot{K} < 0$. When $\dot{K} < 0$, the region $z < z_o$ is the region of loading and $z > z_o$ that of unloading. Consequently, equation (B1) is now valid for $z \geqq z_o$ while equations (B6) and (B8) hold for $z \leqq z_o$. Thus $\dot{N}_x - \nu\dot{N}_y$ becomes

$$\dot{N}_x - \nu\dot{N}_y = \int_{-h/2}^{h/2} \dot{\epsilon}_x \, dz + 2c\dot{K}\int_{-h/2}^{z_o} \left(z - z_o\right) dz \tag{B17}$$

which can be evaluated, as in the previous case, in the form

$$\begin{aligned} \dot{N}_x - \nu\dot{N}_y &= \dot{\epsilon}_1 h - c\dot{K}\left(-\frac{h}{2} - z_o\right)^2 \\ &= \dot{\epsilon}_1 h - c\dot{K}\left(\frac{h}{2} + z_o\right)^2 \end{aligned} \tag{B18}$$

Similarly,

$$\dot{N}_y - \nu\dot{N}_x = \dot{\epsilon}_2 h + \frac{c}{2}\dot{K}\left(\frac{h}{2} + z_o\right)^2 \tag{B19}$$

Again the rates of the reduced stress resultants $\dot{N}_x$ and $\dot{N}_y$ must vanish. This gives

$$\left(\frac{h}{2} + z_o\right)^2 = \frac{\dot{\epsilon}_1 h}{c\dot{K}} = -\frac{2\dot{\epsilon}_2 h}{c\dot{K}} \tag{B20}$$

Equations (49) can be used to show that

$$z_o = (2 - \nu)\frac{\dot{\epsilon}_1}{\dot{K}} + (2\nu - 1)\frac{\dot{\epsilon}_2}{\dot{K}}$$

$$= -\frac{\dot{\epsilon}_2}{\dot{K}}(5 - 4\nu) \qquad \text{(B21)}$$

and equations (B20) and (B21) yield

$$\left(\frac{h}{2} + z_o\right)^2 = -\frac{2\dot{\epsilon}_2 h}{c\dot{K}} = \frac{2z_o h}{c(5 - 4\nu)} \qquad \text{(B22)}$$

Equation (B22) is the analogue of equation (58) which was found for the case $\dot{K} > 0$. It can also be solved by introducing the nondimensional quantity $\zeta_o$ defined as $\zeta_o = 2z_o/h$. The quadratic equation for $\zeta_o$ is

$$\left(1 + \zeta_o\right)^2 - \frac{4\zeta_o}{c(5 - 4\nu)} = 0 \qquad \text{(B23)}$$

and this can be transformed into

$$\zeta_o^2 + 2\left[1 - \frac{2}{c(5 - 4\nu)}\right] + 1 = 0 \qquad \text{(B24)}$$

The quantity appearing in the brackets was defined as $\alpha$ in equations (64); thus the solution of equation (B24) is given by

$$\zeta_o = -\alpha \pm \sqrt{\alpha^2 - 1} \qquad \text{(B25)}$$

Again, the restriction that $|\zeta_o| \leqq 1$ requires that only one of the signs in equation (B25) can be taken, namely, the minus sign. Thus the solution is that given by equation (63); namely,

$$\zeta_o = -\alpha - \sqrt{\alpha^2 - 1} = \zeta_o^- \qquad \text{(B26)}$$

where the symbol $\zeta_o^-$ has been introduced to distinguish this case from the root $\zeta_o^+$ previously found. Comparison of equations (B16) and (B26) shows that

$$\zeta_o^- = -\zeta_o^+ \tag{B27}$$

In addition, $\zeta_o^+ < 0$ for $\dot{K} > 0$ and $\zeta_o^- > 0$ for $\dot{K} < 0$, as might be expected from the geometry of the situation.

## Determination of Rates of Change of Bending and Twisting Moments

The moment rates can be computed now that $z_o$ or $\zeta_o$ is known for each of the cases $\dot{K} > 0$ and $\dot{K} < 0$. Since the rates of change of the reduced bending moments $\dot{M}_x$ and $\dot{M}_y$ are defined as in equation (64), the device of computing $\dot{M}_x - \nu\dot{M}_y$ and $\dot{M}_y - \nu\dot{M}_x$ may be used here.

Case (1), $\dot{K} > 0$.- Application of equations (B1) and (B6) yields

$$\dot{M}_x - \nu\dot{M}_y = \int_{-h/2}^{h/2} \dot{\epsilon}_x z \, dz + 2c\dot{K}\int_{z_o}^{h/2} z(z - z_o) \, dz$$

$$= -\frac{\dot{K}_1 h^3}{12} + 2c\dot{K}\left(\frac{h^3}{24} - \frac{z_o^3}{3} - \frac{h^2 z_o}{8} + \frac{z_o^3}{2}\right) \tag{B29}$$

The quantity $z_o$ can be replaced by $h\zeta_o^+/2 = z_o$ with the result

$$\dot{M}_x - \nu\dot{M}_y = -\frac{h^3}{12}\left[\dot{K}_1 - c\dot{K}\left(1 - \frac{3}{2}\zeta_o^+ + \frac{1}{2}\zeta_o^{+3}\right)\right] \tag{B30}$$

With

$$2\delta = 1 - \frac{3}{2}\zeta_o^{+} + \frac{1}{2}\zeta_o^{+3} \tag{B31}$$

the last equation becomes

$$\dot{M}_x - \nu\dot{M}_y = -\frac{h^3}{12}\left(\dot{K}_1 - 2c\delta\dot{K}\right) \tag{B32}$$

Equation (B8) for $\dot{\sigma}_y - \nu\dot{\sigma}_x$ differs from equation (B6) for $\dot{\sigma}_x - \nu\dot{\sigma}_y$ only in that $-\dot{K}$ replaces $2\dot{K}$ and $\dot{\epsilon}_y$ replaces $\dot{\epsilon}_x$ in equation (B6). Without further computation, it can be seen that

$$\dot{M}_y - \nu\dot{M}_x = -\frac{h^3}{12}\left(\dot{K}_2 + c\delta\dot{K}\right) \tag{B33}$$

Equations (B32) and (B33) can be solved for the reduced bending moments with the result that

$$\dot{M}_x = -\frac{h^3}{12\left(1 - \nu^2\right)}\left[\dot{K}_1 + \nu\dot{K}_2 - (2 - \nu)c\delta\dot{K}\right] \tag{B34}$$

$$\dot{M}_y = -\frac{h^3}{12\left(1 - \nu^2\right)}\left[\nu\dot{K}_1 + \dot{K}_2 - (2\nu - 1)c\delta\dot{K}\right] \tag{B35}$$

The curvature $\dot{K}$ can be expressed in terms of $\dot{K}_1$ and $\dot{K}_2$ according to equations (49); consequently,

$$\dot{M}_x = -\frac{h^3}{12(1 - \nu^2)}\left\{\dot{K}_1\left[1 - c\delta(2 - \nu)^2\right] + \dot{K}_2\left[\nu - c\delta(2 - \nu)(2\nu - 1)\right]\right\} \tag{B36}$$

$$\dot{M}_y = -\frac{h^3}{12(1 - \nu^2)}\left\{\dot{K}_1\left[\nu - c\delta(2 - \nu)(2\nu - 1)\right] + \dot{K}_2\left[1 - c\delta(2\nu - 1)^2\right]\right\} \tag{B37}$$

These relations are the same as equations (67) and (68) which were given in the second section of the ANALYSIS without proof.

Case (2), $\dot{K} < 0$.- This case can be handled in exactly the same manner as the previous one provided the integrals are split up as follows:

$$\dot{M}_x - \nu\dot{M}_y = \int_{-h/2}^{h/2} \dot{\epsilon}_x z\, dz + 2c\dot{K}\int_{-h/2}^{z_o} z\left(z - z_o\right) dz \tag{B38}$$

$$= -\frac{\dot{K}_1 h^3}{12} + 2c\dot{K}\left(\frac{z_o^3}{3} + \frac{h^3}{24} - \frac{z_o^3}{2} + \frac{z_o h^2}{8}\right) \tag{B39}$$

For $\dot{K} < 0$, $z_o = h\zeta_o^- / 2$ and

$$\dot{M}_x - \nu\dot{M}_y = -\frac{h^3}{12}\left[\dot{K}_1 - c\dot{K}\left(1 - \frac{1}{2}\zeta_o^{-3} + \frac{3}{2}\zeta_o^-\right)\right] \tag{B40}$$

Now $\zeta_o^- = -\zeta_o^+$; thus,

$$\left(1 - \frac{1}{2}\zeta_o^{-3} + \frac{3}{2}\zeta_o^-\right) = \left(1 + \frac{1}{2}\zeta_o^{+3} - \frac{3}{2}\zeta_o^+\right) = 2\delta \tag{B41}$$

In other words, $\dot{M}_x - \nu\dot{M}_y$ will be given by the same formula for $\dot{K} < 0$ as for $\dot{K} > 0$. Similarly $\dot{M}_y - \nu\dot{M}_x$ will be given by the same expression in either case; and, consequently, $\dot{M}_x$ and $\dot{M}_y$ can be found from equations (B36) and (B37) independently of whether $\dot{K} < 0$ or $\dot{K} > 0$.

The rate of change of the reduced twisting moment $\dot{M}_{xy}$ is defined in equation (65). According to equations (40) and (45) equation (65) may be rewritten as

$$\dot{M}_{xy} = -\frac{-1}{2(1 + \nu)}\int_{-h/2}^{h/2} \dot{\gamma}_{xy} z \, dz \tag{B42}$$

$$= -\frac{1}{2(1 + \nu)}\int_{-h/2}^{h/2} \left(\dot{\gamma} - z\dot{K}_{12}\right) z \, dz \tag{B43}$$

Integration of the last equation furnishes

$$\dot{M}_{xy} = \frac{h^3}{12\left(1 - \nu^2\right)} \frac{(1 - \nu)}{2} \dot{K}_{12} \tag{B44}$$

This result, too, holds independently of whether $\dot{K} > 0$ or $\dot{K} < 0$.

# APPENDIX C

## DETAILED COMPUTATIONS FOR SPECIFIC EXAMPLES

### Buckling of a Thin Strip

According to the definitions of $r$ and $s$, equations (85), and of $p$, equations (82),

$$r^2 - s^2 = 2p^2 = 2\left(\frac{m\pi}{a}\right)^2 \frac{D_{12}}{D_{22}} \tag{C1}$$

The terms of the second row of the determinant (equation (90)) can therefore be written in the form

$$\left.\begin{aligned} & r\left\{D_{22}s^2 + \frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\right\} \sinh\frac{rb}{2} \\ & s\left\{D_{22}r^2 - \frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\right\} \sin\frac{sb}{2} \end{aligned}\right\} \tag{C2}$$

The determinantal equation (90) is thus seen to be equivalent to

$$\begin{aligned} & s\left\{D_{22}r^2 - \frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\right\}^2 \tan\frac{sb}{2} \\ & \qquad + r\left\{D_{22}s^2 + \frac{m^2\pi^2}{a^2}\left[D_{12} - (1 - \nu)D\right]\right\}^2 \tanh\frac{rb}{2} = 0 \end{aligned} \tag{C3}$$

With the definitions of equations (92), equation (C3) can be rewritten in the form of equation (91). The evaluation of the roots of the transcendental equation (91) can be simplified further by introducing equations (94). A simple computation will show that

$$\left.\begin{aligned}\xi^2 &= \frac{m^2\pi^2R^2}{D_{22}'}\left(\sqrt{\frac{kD_{22}'}{m^2} - \Delta D_{22}'^2} + D_{12}'\right)\\ \eta^2 &= \frac{m^2\pi^2R^2}{D_{22}'}\left(\sqrt{\frac{kD_{22}'}{m^2} - \Delta D_{22}'^2} - D_{12}'\right)\end{aligned}\right\} \tag{C4}$$

If, for a fixed value of the span $a$, the width $b$ of the plate approaches zero, $R$ and hence $\xi$ and $\eta$ tend toward zero too. Accordingly, the functions $\xi \tanh(\xi/2)$ and $\eta \tan(\eta/2)$ appearing in equations (C4) may be replaced by $\xi^2/2$ and $\eta^2/2$, respectively. For $m = 1$, in particular, equation (91) can then be written as follows:

$$\left(\sqrt{kD_{22}' - \Delta D_{22}'^2} + D_{12}' - QD_{22}'\right)^2\left(\sqrt{kD_{22}' - \Delta D_{22}'^2} - D_{12}'\right)$$
$$+ \left(\sqrt{kD_{22}' - \Delta D_{22}'^2} - D_{12}' + QD_{22}'\right)^2\left(\sqrt{kD_{22}' - \Delta D_{22}'^2} + D_{12}'\right) = 0 \tag{C5}$$

or in the form of equation (93) as

$$kD_{22}' - \Delta D_{22}'^2 - D_{12}'^2 + Q^2D_{22}'^2 = 0$$

Equation (99) for the buckling load when $\nu = \frac{1}{2}$ can be developed from the following considerations. According to equations (73) and (94),

$$\left.\begin{aligned} D_{11}' &= -1 - \frac{9}{4}c\delta \\ D_{12}' &= D_{22}' = 1 \end{aligned}\right\} \tag{C6}$$

for $\nu = \frac{1}{2}$. From these results and equations (92), it is easily seen that $Q = \nu$; consequently,

$$k = \frac{3}{4}(1 - 3c\delta) \tag{C7}$$

when $\nu = \frac{1}{2}$. For this value of Poisson's ratio, a straightforward computation will yield

$$\left.\begin{aligned} c &= \frac{\lambda - 1}{3\lambda} \\ \alpha &= -\frac{\lambda + 1}{\lambda - 1} \\ \zeta_o^+ &= -\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} \\ \delta &= \frac{\lambda\left(\lambda + \sqrt{3}\right)}{\left(1 + \sqrt{\lambda}\right)^3} \end{aligned}\right\} \tag{C8}$$

Therefore,

$$c\delta = \frac{\left(\sqrt{\lambda} - 1\right)\left(\sqrt{\lambda} + 3\right)}{3\left(1 + \sqrt{\lambda}\right)^2} \tag{C9}$$

and

$$k = \frac{3}{4}(1 - 3c\delta) = \frac{3}{\left(1 + \sqrt{\lambda}\right)^2} \tag{C10}$$

On the other hand, according to equations (94),

$$k = \frac{9\sigma_o a^2}{\pi^2 h^2 E_o} \qquad \text{(C11)}$$

where $\sigma_{cr}$ is the critical compressive stress. Consequently,

$$\sigma_{cr} = \frac{\pi^2 E_o h^2}{3c^2} \frac{1}{\left(1 + \sqrt{\lambda}\right)^2} \qquad \text{(C12)}$$

and the critical buckling load $P$ is given by

$$P = \sigma_{cr} bh = \frac{\pi^2 E_o I}{c^2} \frac{4}{\left(1 + \sqrt{\lambda}\right)^2}$$

from equation (99). This is the desired result from which Von Kármán's equation follows.

It has been pointed out in the third section of the ANALYSIS that the neutral line of the beam and the neutral surface of the plate will coincide if and only if $\nu = \frac{1}{2}$. This can be seen quite simply from the following discussion. If $\nu = \frac{1}{2}$, then $z_o = \dot{\epsilon}_1/\dot{K}$ according to equations (49). It follows from equations (C8) that

$$\zeta_o^+ = \frac{2z_o}{h} = -\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} \qquad \text{(C13)}$$

The distance of the neutral surface from the lower surface is given by

$$\frac{h}{2} + z_o = \frac{h}{\sqrt{\lambda} + 1} = h \frac{\sqrt{E}}{\sqrt{E_o} + \sqrt{E}} \qquad \text{(C14)}$$

This is the same result as that obtained in the direct analysis of the buckling of a beam. (See reference 1, p. 158.)

Conversely, assume that the two positions of the neutral line agree; that is,

$$\zeta_o^{+} = -\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} \tag{C15}$$

where $\zeta_o^{+}$ is computed from equation (B16). Since equation (C15) must be an identity in $\lambda$, it must be valid for any particular value of $\lambda$. For computational convenience, consider the case when $\lambda = 4$. Then

$$\left.\begin{aligned} -\frac{\sqrt{\lambda} - 1}{\sqrt{\lambda} + 1} &= -\frac{1}{3} \\ c &= \frac{3}{19 - 12\nu - 4\nu^2} \\ \alpha &= \frac{u}{3(5 - 4\nu)} \end{aligned}\right\} \tag{C16}$$

where $u = 8\nu^2 + 12\nu - 23$. Consequently,

$$\begin{aligned} \zeta_o^{+} &= \frac{u}{3(5 - 4\nu)} + \frac{1}{3(5 - 4\nu)}\sqrt{u^2 - 9(5 - 4\nu)^2} \\ &= -\frac{1}{3} \end{aligned} \tag{C17}$$

and subsequent simplification leads to

$$u = -5(5 - 4\nu) \tag{C18}$$

Evaluation of $u$ in terms of $\nu$ yields

$$(2\nu - 1)^2 = 0 \tag{C19}$$

or

$$\nu = \frac{1}{2} \tag{C20}$$

Therefore the two methods of determining the neutral line will agree if and only if $\nu = \frac{1}{2}$, and the transition from plate to beam will be valid only under this condition.

## Buckling of a Cruciform Section

The transcendental equation (138) must be solved to determine the critical stress. A straightforward computation will show that

$$\left.\begin{aligned} \xi^2 &= \frac{m^2\pi^2R^2}{D_{22}'}\left(\sqrt{\frac{kD_{22}'}{m^2} - \Delta D_{22}'^2} + D_{12}'\right) \\ \eta^2 &= \frac{m^2\pi^2R^2}{D_{22}'}\left(\sqrt{\frac{kD_{22}'}{m^2} - \Delta D_{22}'^2} - D_{12}'\right) \end{aligned}\right\} \tag{C21}$$

Since

$$\xi^2 = \eta^2 + \frac{2m^2\pi^2R^2D_{12}'}{D_{22}'} \tag{C22}$$

Equation (138) can be written entirely in terms of $\eta$ and the parameters $R$, $Q$, $D_{12}'$, and $D_{22}'$. For given values of $m$ and $R$, the remaining parameters in the transcendental equation are functions of $\lambda$ alone. Thus $\eta$ can be determined as a function of $\lambda$ by solving this equation by the usual iterative procedures. The quantity $k$ can then be found as a function of $\lambda$ by solving the second of equations (C21). These results are independent of the stress-strain law of the plate material. On the other hand,

$$k = \frac{12\sigma_o a^2}{\pi^2 h^2 E_o}\left(1 - \nu^2\right) \tag{C23}$$

that is, for a given material $k$ can be written as a function of $\lambda$ once $\sigma_o$ is known as a function of $\lambda$.

It is a little more convenient to write the solution in terms of the quantity $\bar{k}$ rather than $k$. This parameter was defined in equation (117). Curves for $\bar{k}$ as a function of $\lambda$ for given values of $m$ and $R$ can be obtained in the following way. The quantity $k$ can be computed, as previously described, by solving equations (138) and (C21); $\bar{k}$ can then be found from equation (117). These results are independent of the stress-strain law and consequently hold for any rectangular plate. On the other hand, $\bar{k}$ can also be computed as a function of $\lambda$ for a given stress-strain law according to equation (117). The resulting function will depend on the parameter $b^2/h^2$. Once the side ratio $R$, the width-to-thickness ratio $b/h$, and the wave form $m$ have been selected, the value of $\bar{k}$ corresponding to the buckling stress can be obtained by finding the intersection of the two corresponding curves of $\bar{k}$ against $\lambda$ computed as just outlined.

It has been pointed out in the third section of the ANALYSIS that the lowest value of $\bar{k}$, and hence the lowest critical stress, will be attained for $m = 1$. This can be seen easily from the following considerations. Figure 9 showed that for $m = 1$ and a fixed value of $\lambda$, $\bar{k}$ increases for increasing $R$. With a simple change of variables, the solution for any value of $m$ can be obtained from the solutions for $m = 1$. Let

$$\left.\begin{aligned} k' &= \frac{k}{m^2} \\ R' &= mR \\ \bar{k}' &= \frac{\bar{k}}{m^2} \end{aligned}\right\} \tag{C24}$$

Then equations (138) and (C21), written in terms of $k'$ and $R'$ will be precisely the same as those for $k$ and $R$ when $m = 1$. Thus the curves of figure 9 can be used for any $m$ provided $R$ is replaced by $R'$ and $\overline{k}$ by $\overline{k}'$. For $m > 1$ then, $\overline{k} > \overline{k}'$ and $R < R'$. In other words when $m \neq 1$, the value of $\overline{k}$ for a given value of $R$ and $\lambda$ will be larger than the corresponding value of $\overline{k}$ for $m = 1$. Since the dashed curves are monotonic, increasing functions of $\lambda$, this implies that the critical value of $\overline{k}$ will be lowest for $m = 1$.

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TABLE I

VALUES OF $D_{11}'$, $D_{12}'$, AND $D_{22}'$

FOR $\nu = 0.32$

| $\lambda$ | $D_{11}'$ | $D_{12}'$ | $D_{22}'$ |
|---|---|---|---|
| 1.0 | 1.0000 | 1.0000 | 1.0000 |
| 1.2 | .93030 | 1.0149 | .99680 |
| 1.4 | .87447 | 1.0269 | .99424 |
| 1.6 | .82847 | 1.0368 | .99212 |
| 1.8 | .78973 | 1.0451 | .99034 |
| 2.0 | .75655 | 1.0522 | .98882 |
| 2.5 | .69086 | 1.0662 | .98580 |
| 3.0 | .64178 | 1.0768 | .98355 |
| 3.5 | .60345 | 1.0850 | .98179 |
| 4.0 | .57255 | 1.0916 | .98037 |
| 4.5 | .54698 | 1.0971 | .97920 |
| 5.0 | .52544 | 1.1017 | .97821 |
| 6.0 | .49104 | 1.1091 | .97663 |
| 7.0 | .46459 | 1.1147 | .97541 |
| 8.0 | .44365 | 1.1192 | .97445 |
| 9.0 | .42652 | 1.1229 | .97367 |
| 10.0 | .41226 | 1.1259 | .97301 |

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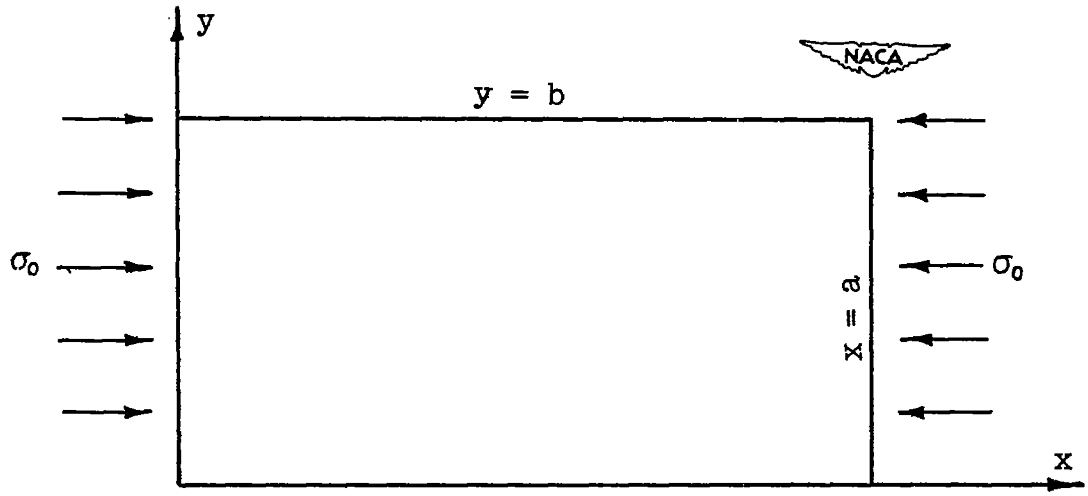



Figure 1.- Plate under uniform compressive stress in direction of x-axis prior to buckling.

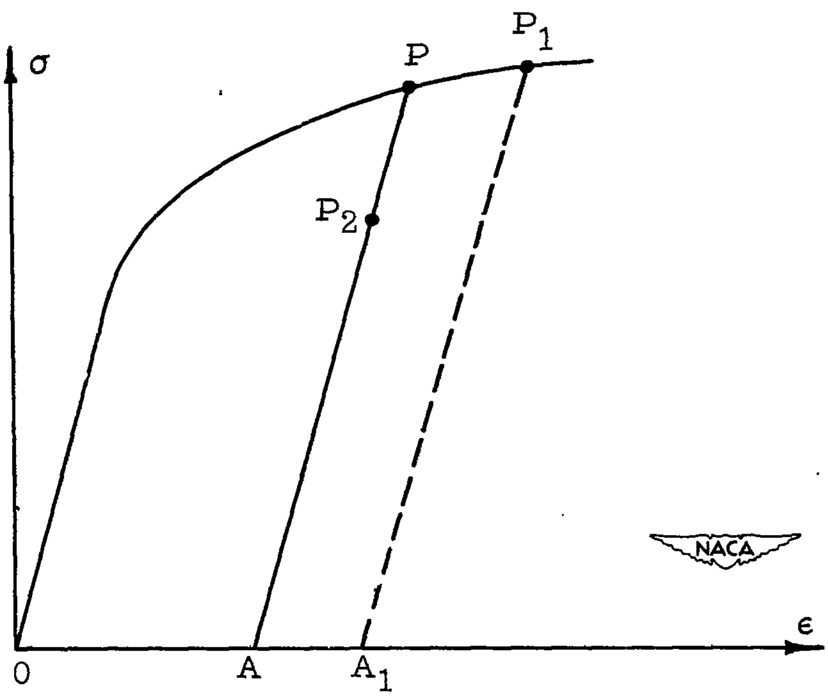



Figure 2.- Stress-strain diagram for loading and unloading for uniaxial state of stress and strain.

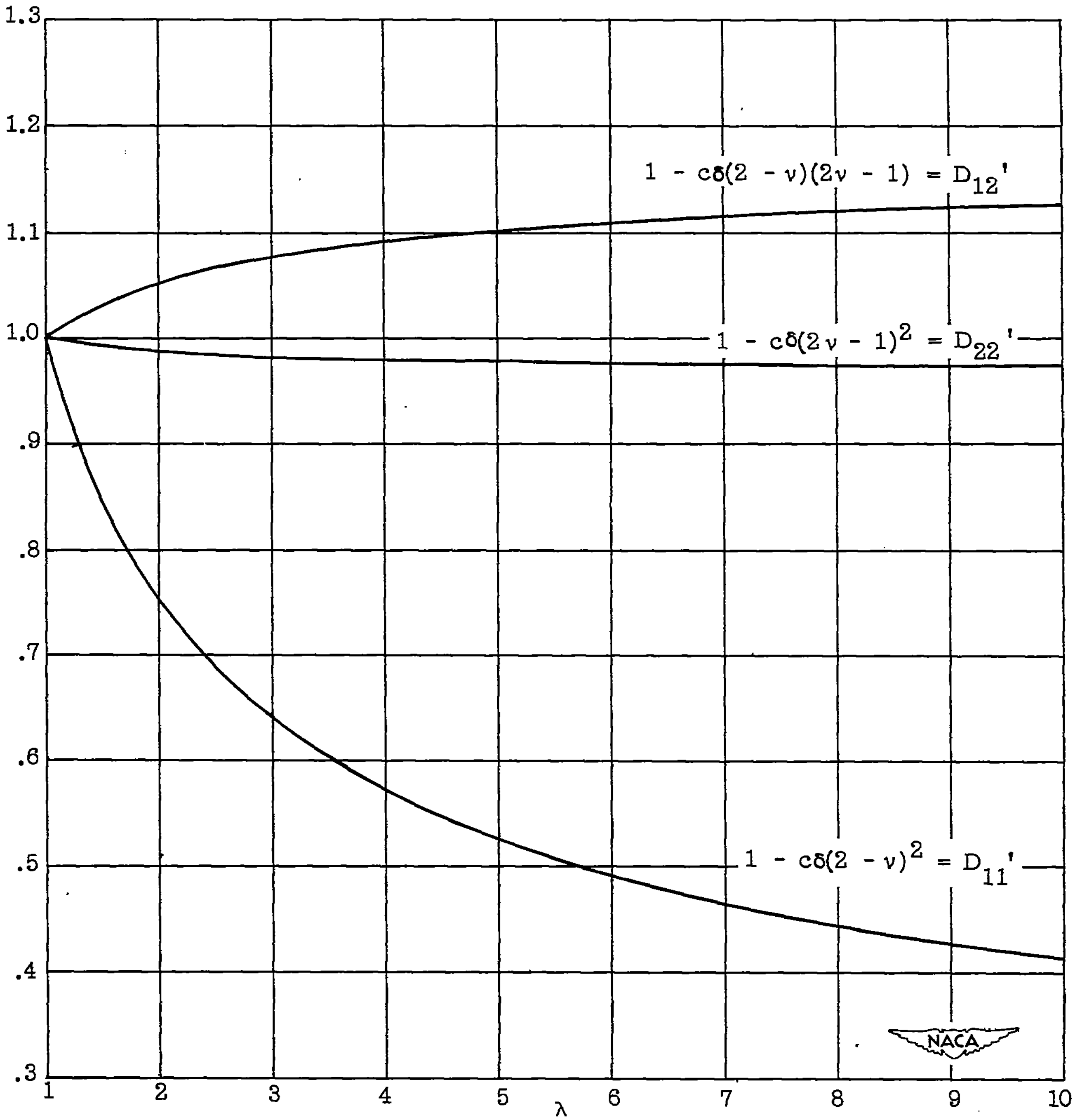



Figure 3.- Graphs of $D_{11}'$, $D_{12}'$, and $D_{22}'$ as functions of $\lambda$ for $\nu = 0.32$.

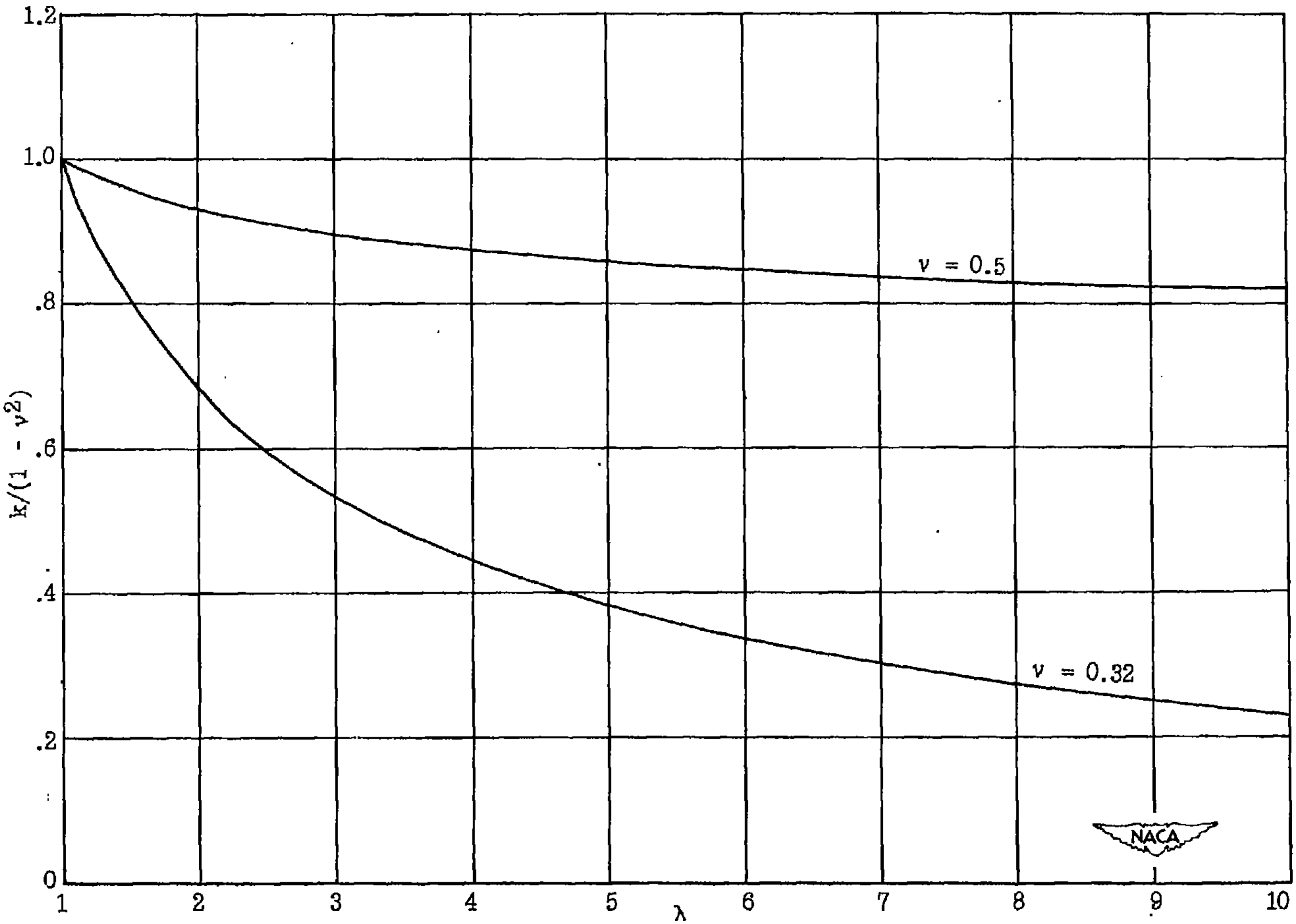



Figure 4.- The quantity $k/\left(1 - \nu^2\right)$ as a function of $\lambda$ for two beams having the same shape but different values of $\nu$.

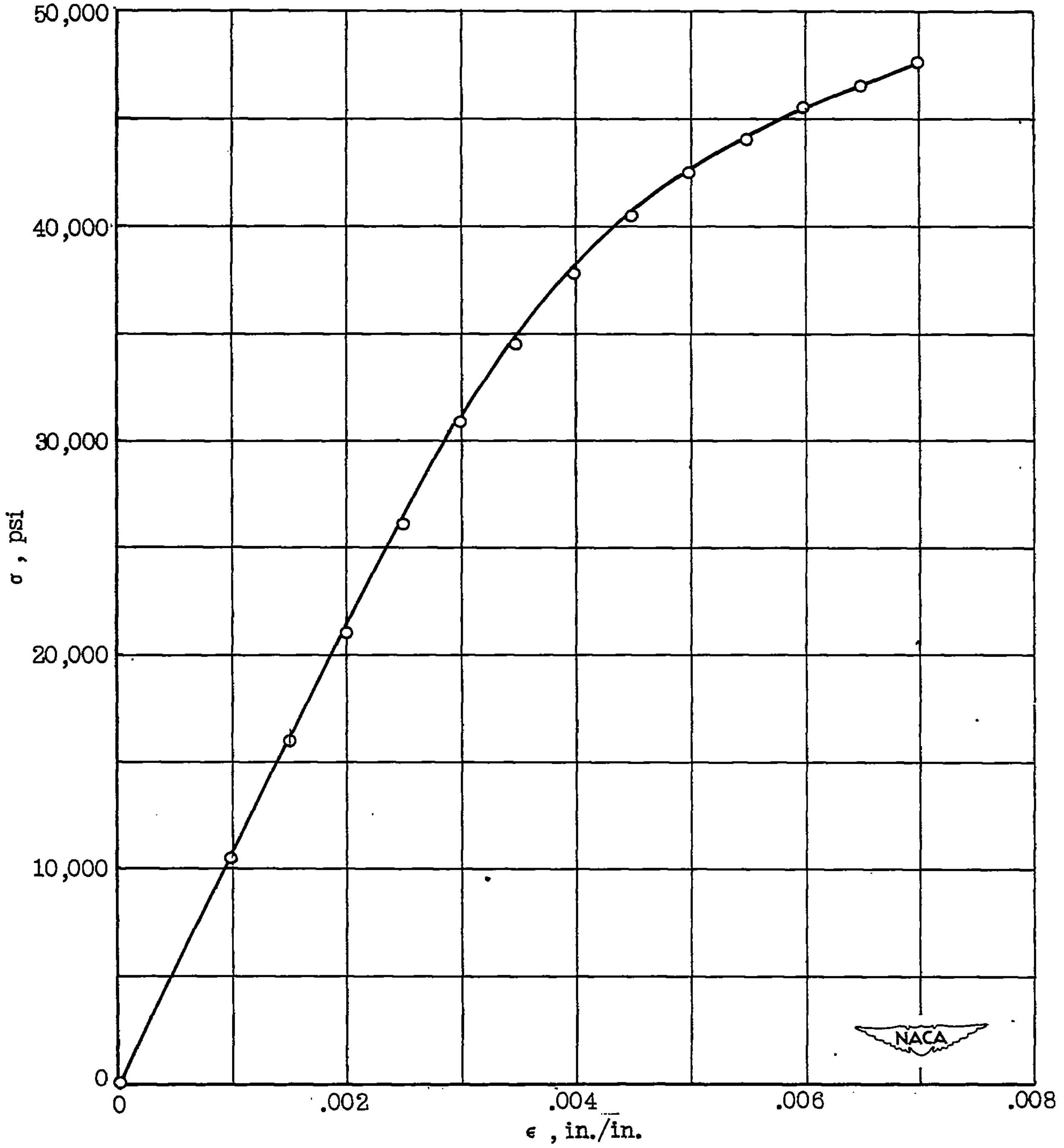
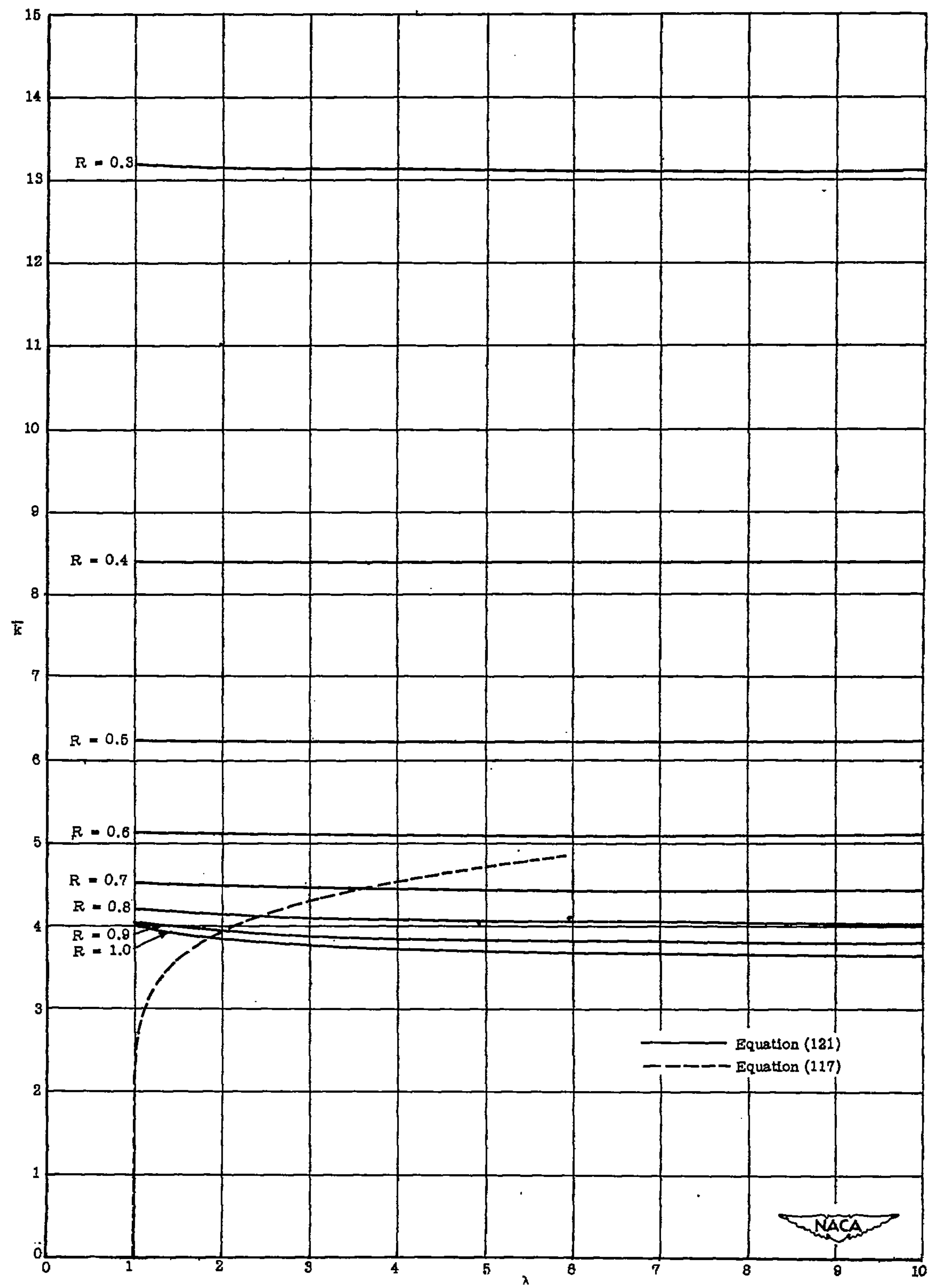
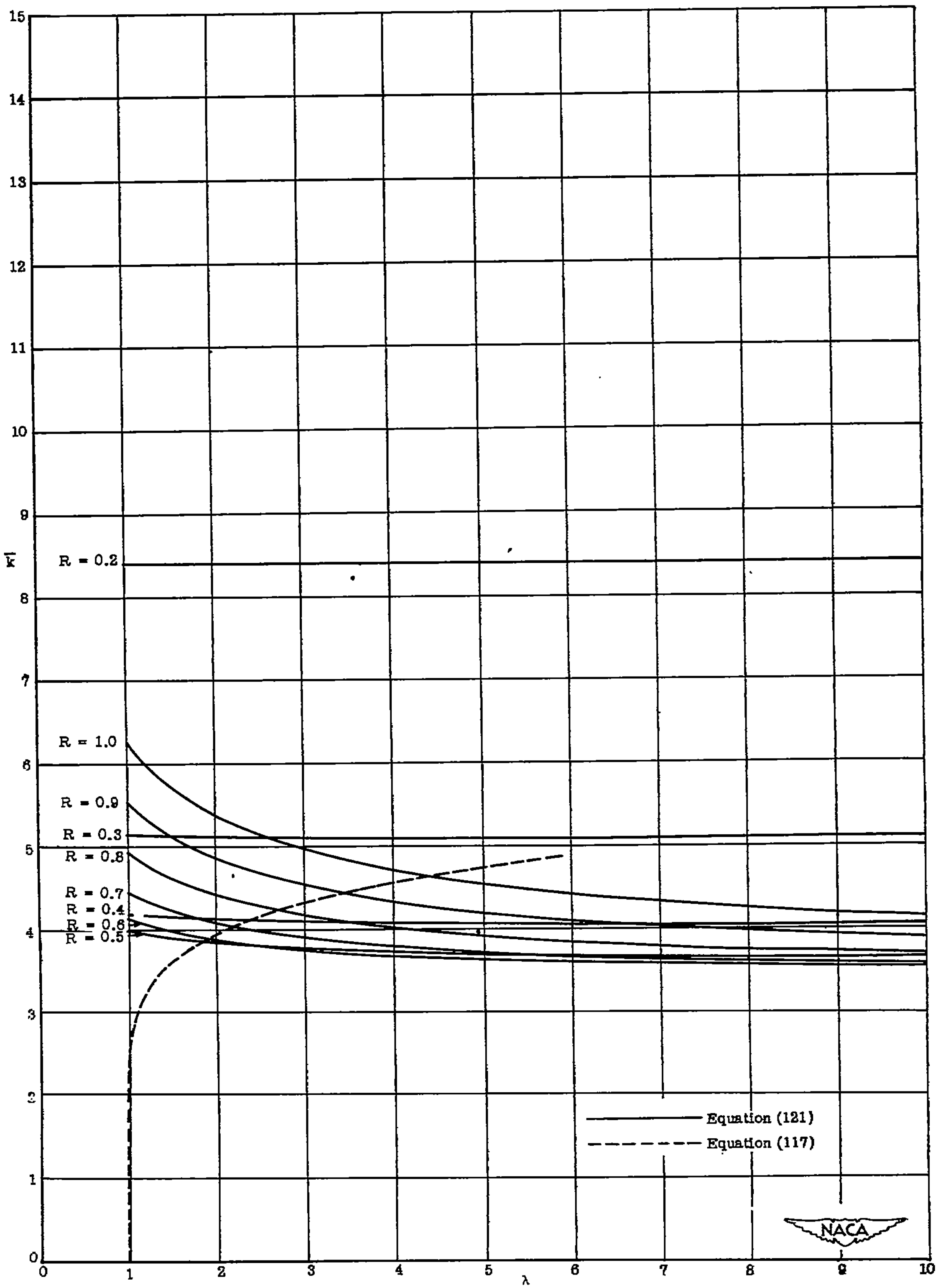



Figure 5.- Stress-strain curve for $\epsilon = \frac{1}{E_o}\sigma + A\sigma^{\beta}$, $E_o = 10,667,000$, $\log A = -42.873$, and $\beta = 8.6127$.



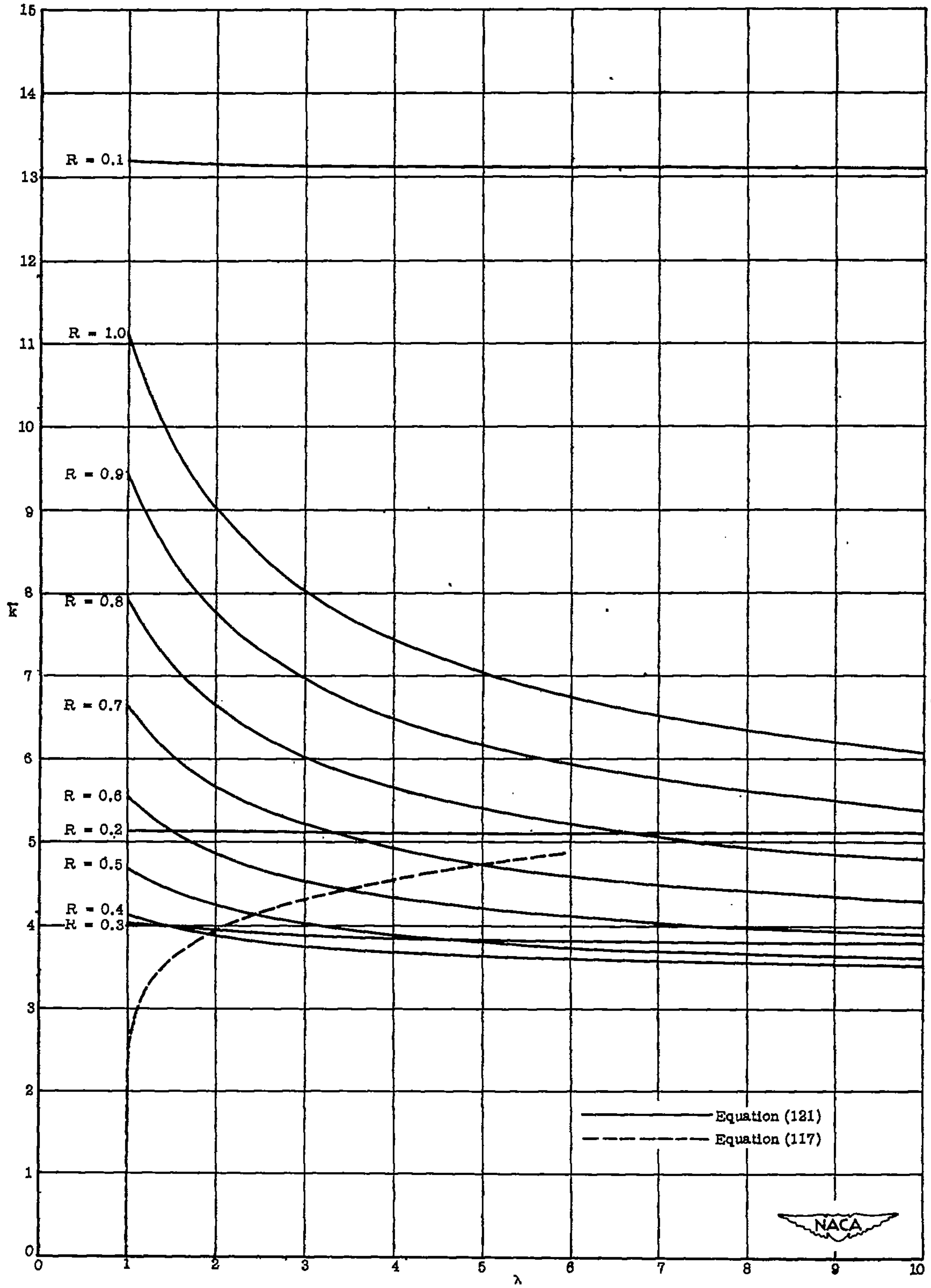

(a) m = 1.

Figure 6.- Curves of $\bar{k}$ as a function of $\lambda$ for different values of m.



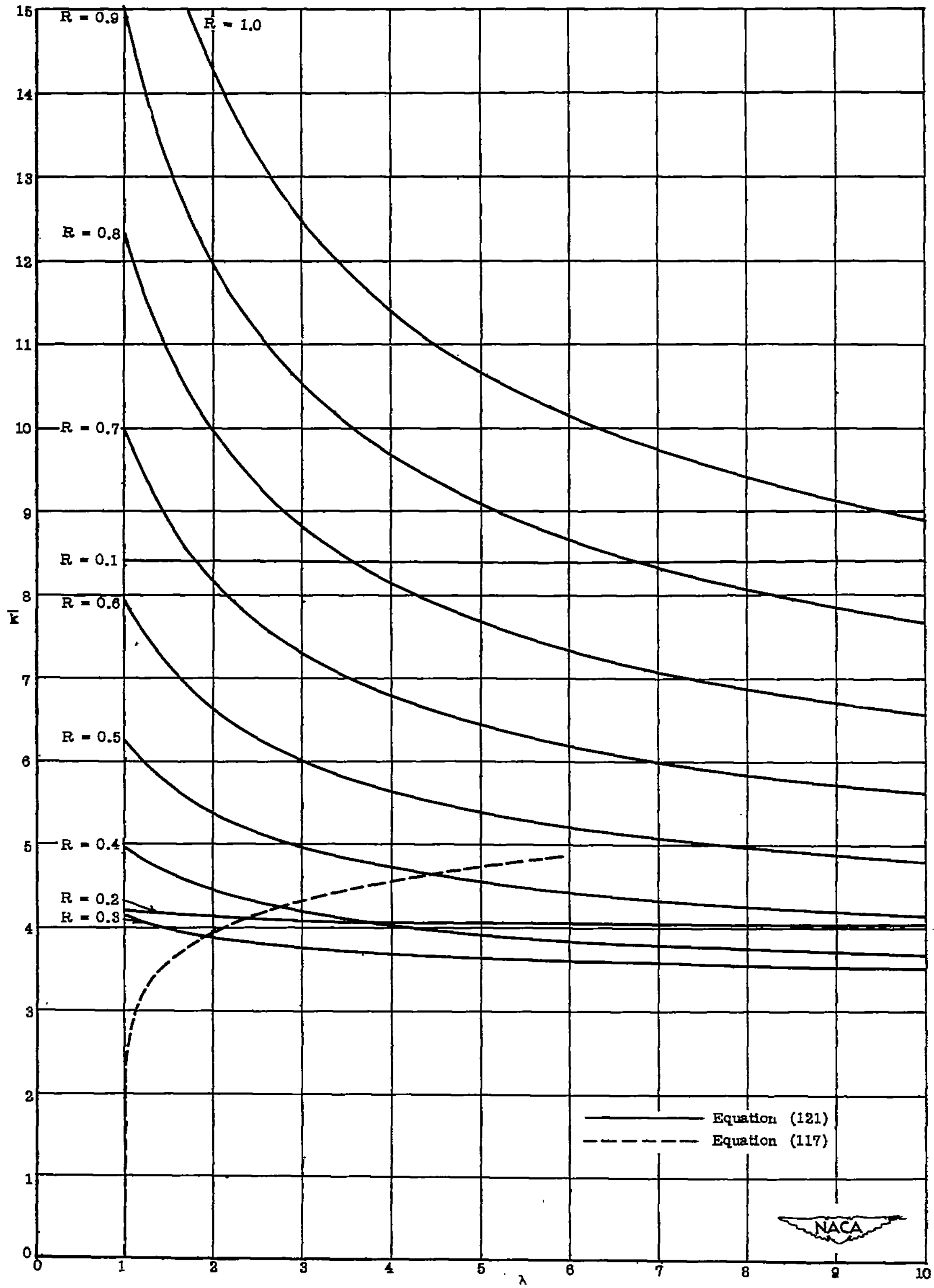

(b) $m = 2$.

Figure 6.- Continued.




(c) m = 3.

Figure 6.- Continued.




(d) m = 4.

Figure 6.- Concluded.

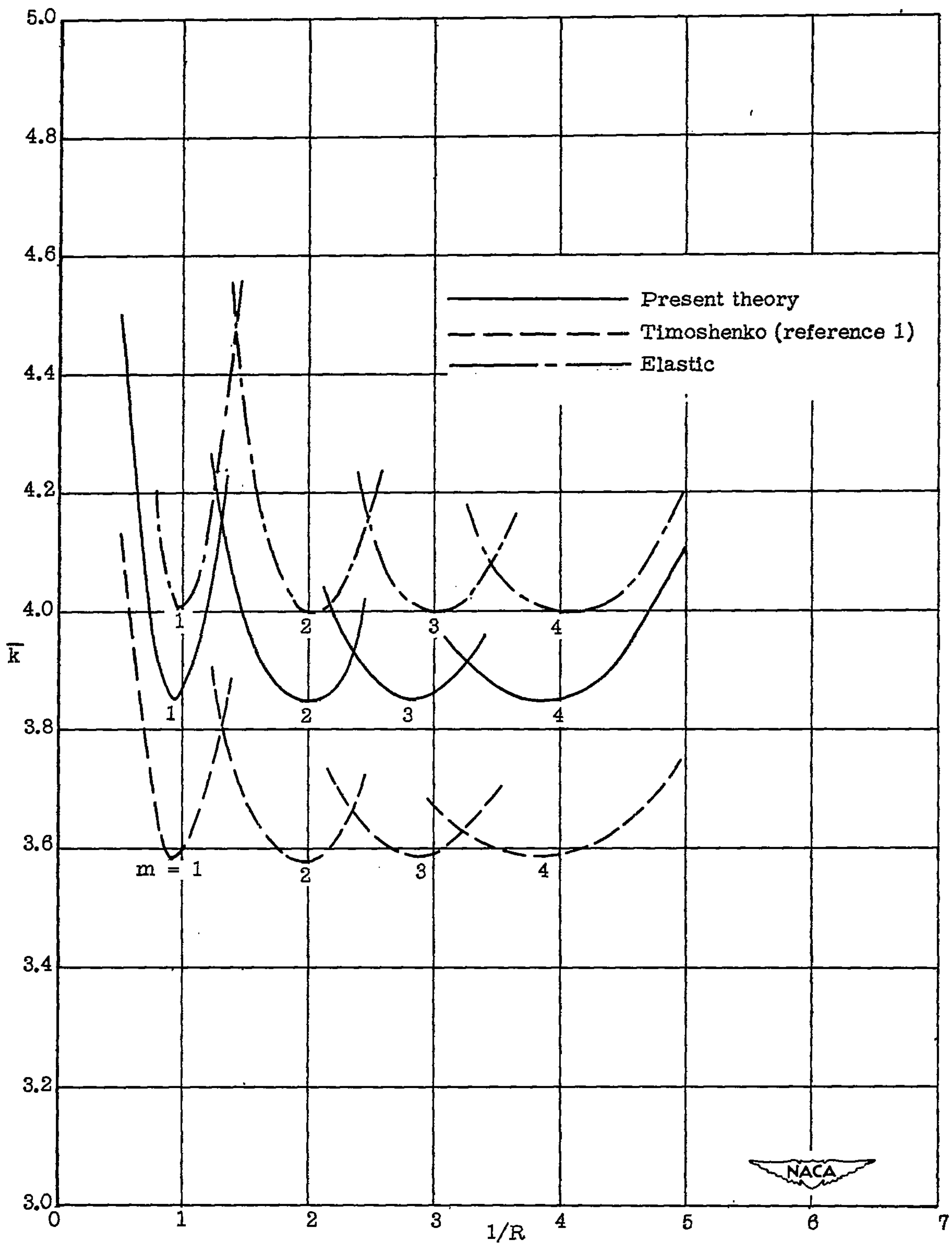



Figure 7.- Comparison of values of $\bar{k}$ for values of $1/R$ with $\nu = 0.32$ as determined by different theories.

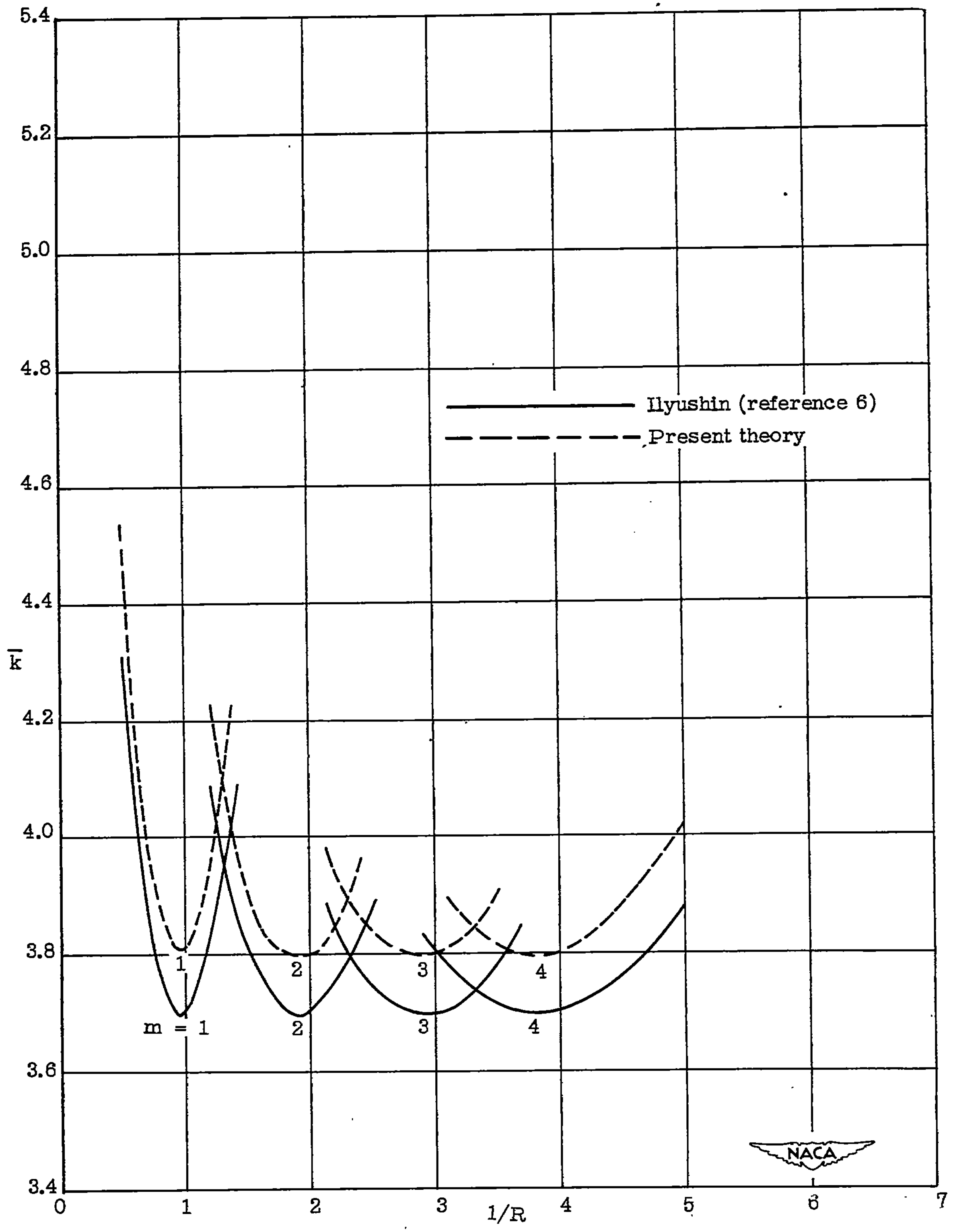



Figure 8.- Comparison of values of $\bar{k}$ for values of $1/R$ with $\nu = 1/2$ as determined by the present method and that of Ilyushin.

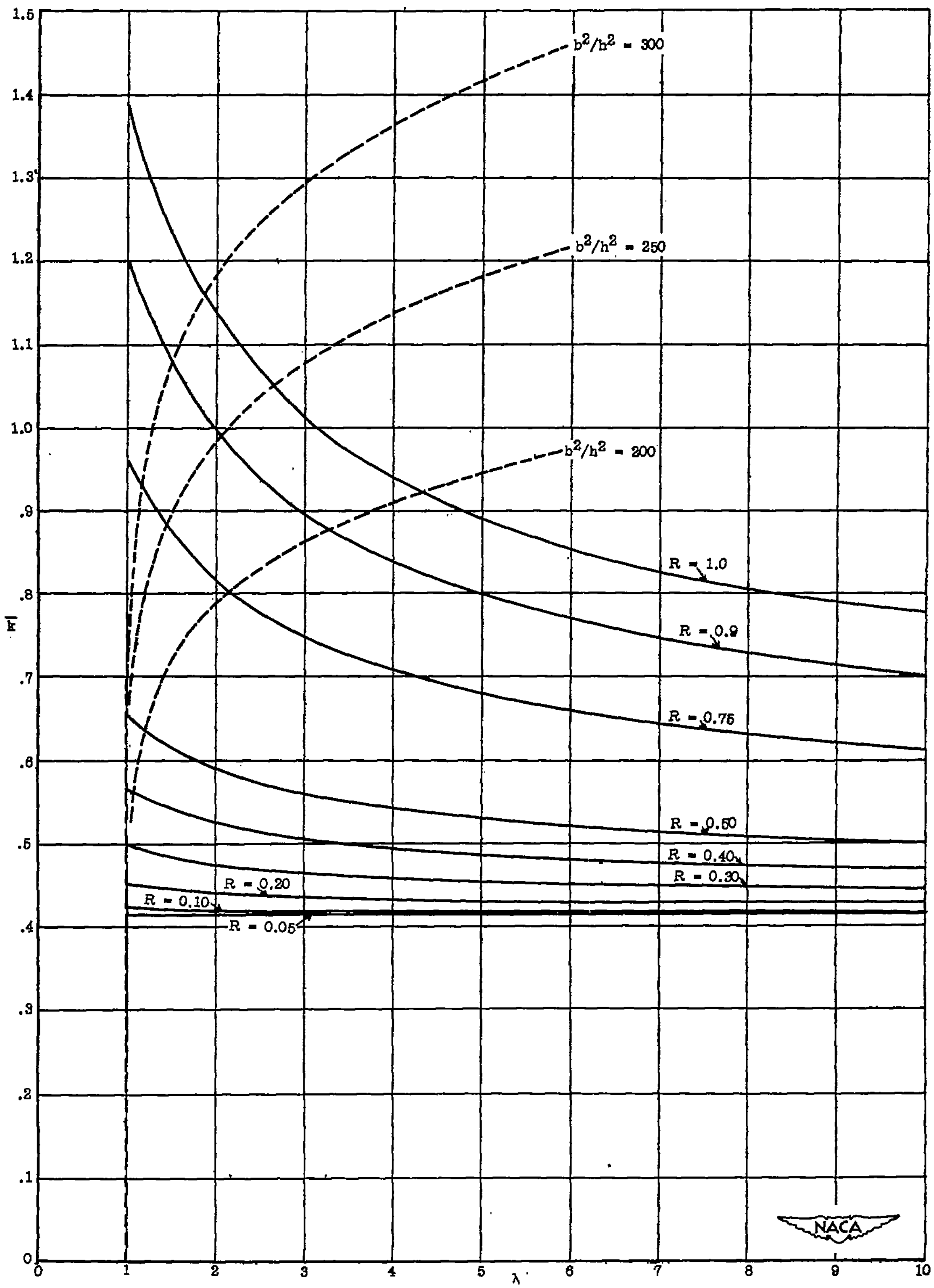



Figure 9.- Results of solving for $\bar{k}$ as a function of $\lambda$ by using equations (117) and (138).

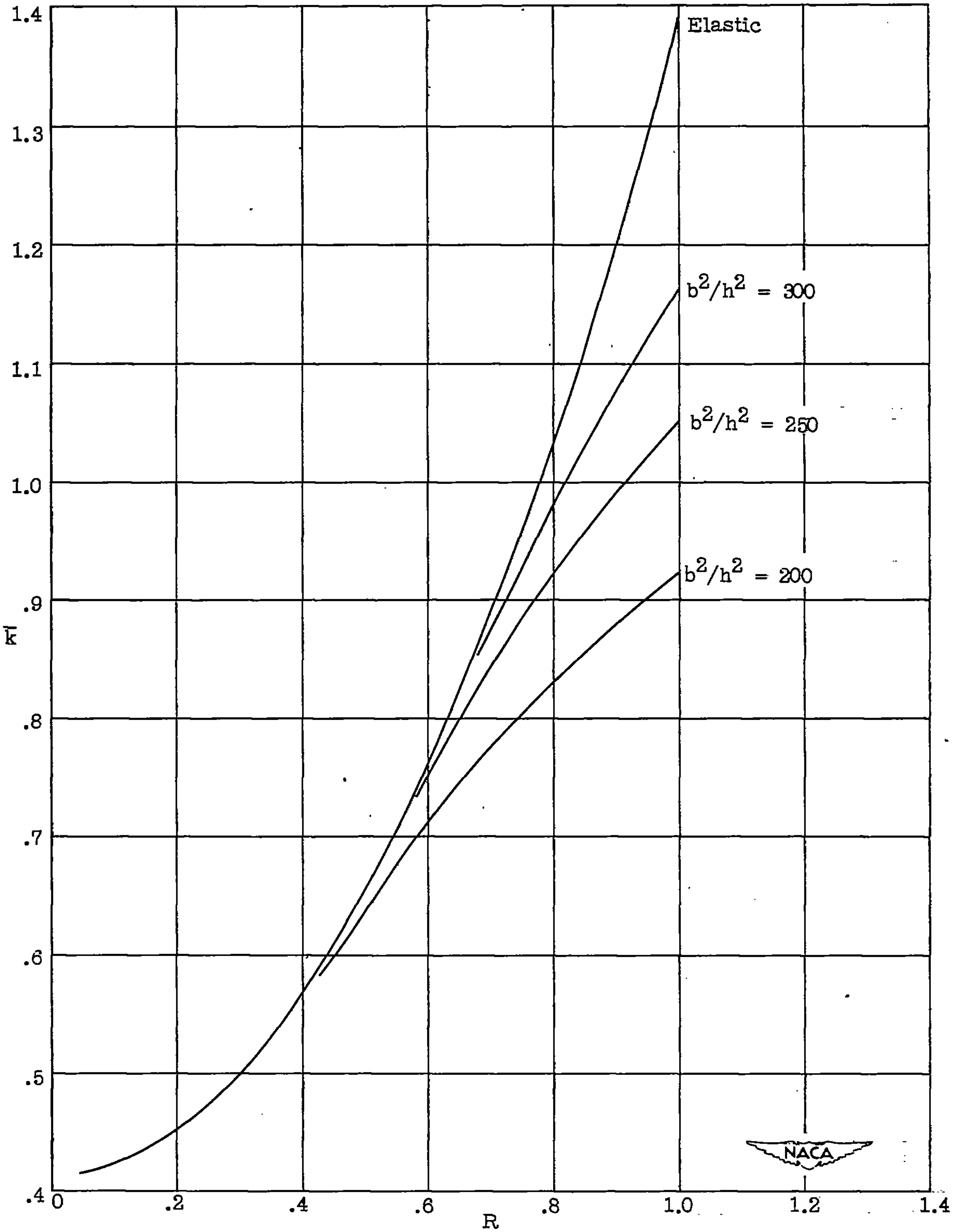



Figure 10.- Curves of $\bar{k}$ as a function of R from figure 9.

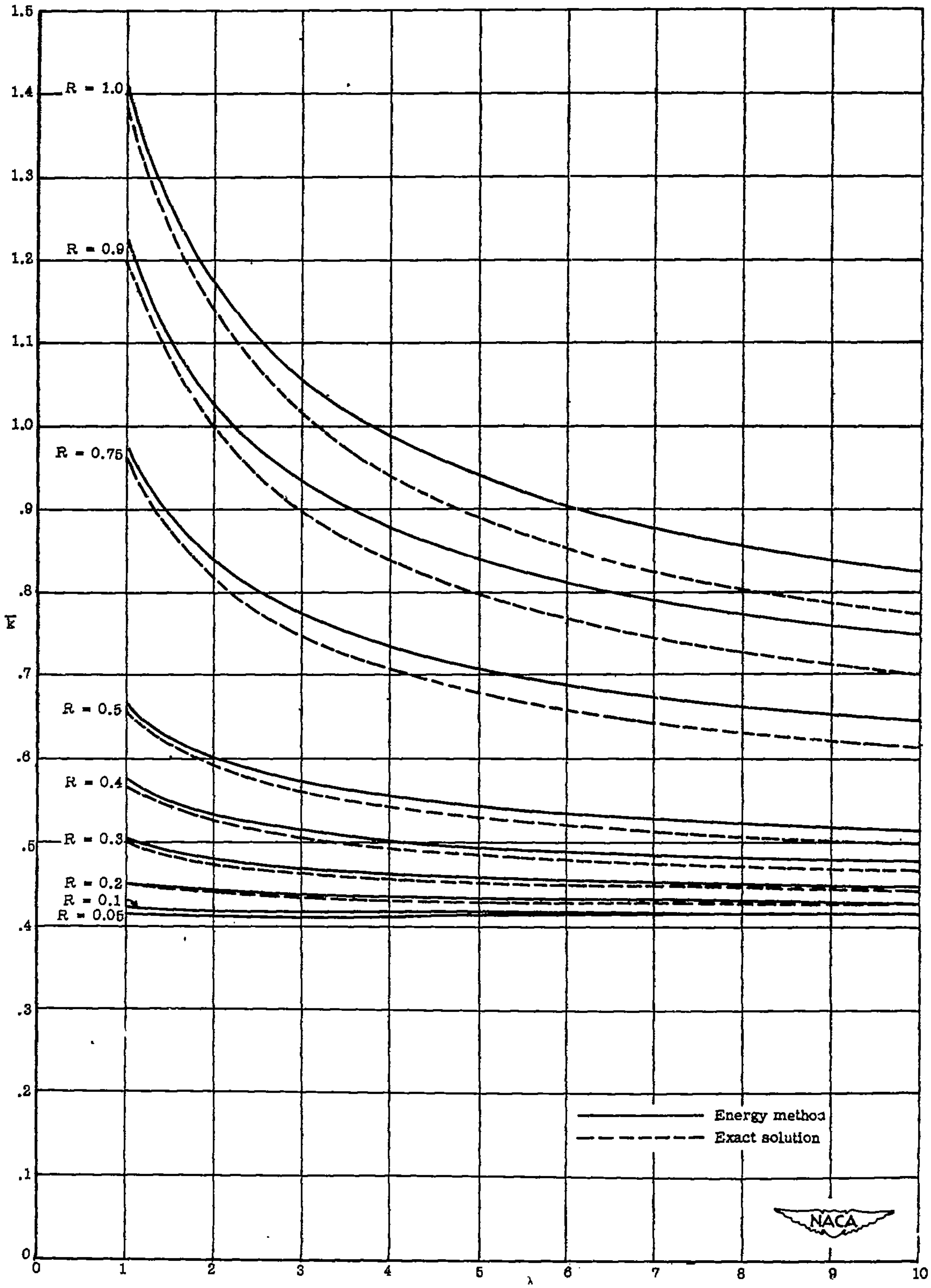



Figure 11.- Curves of $\overline{k}$ obtained by the energy method and exact solution for $\nu = 0.32$ and various values of R.

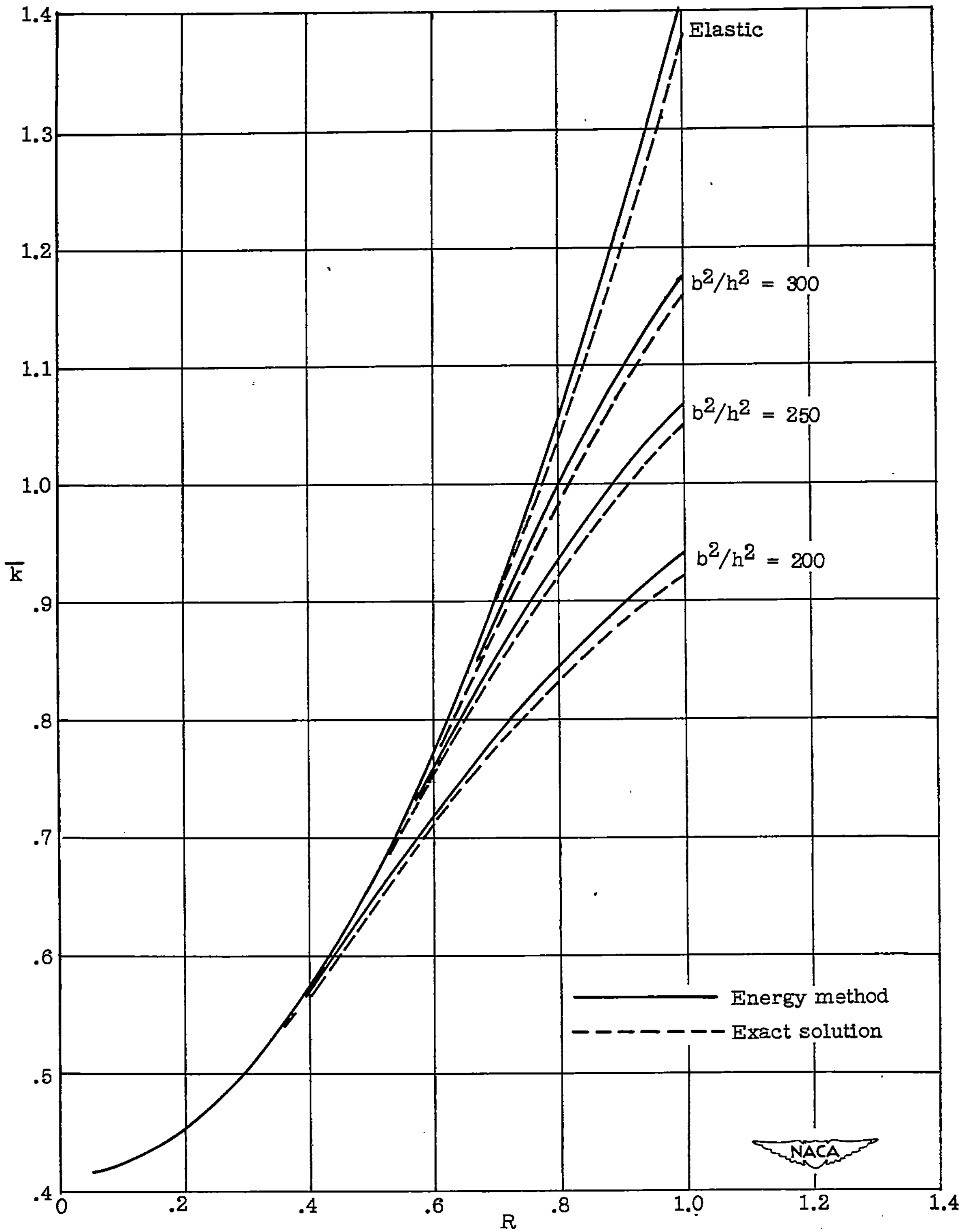



Figure 12.- Results of applying energy method to cases of figure 10.